matics Library

MAY 1 3 1946

# Mathematical Tables
## *and other*
# Aids to Computation

---

A Quarterly Journal edited on behalf of the
Committee on Mathematical Tables
and Other Aids to Computation
by

RAYMOND CLARE ARCHIBALD
DERRICK HENRY LEHMER

WITH THE COÖPERATION OF

LESLIE JOHN COMRIE
SOLOMON ACHILLOVICH JOFFE

---

II • Number 14 • April 1946

*Published by*

THE NATIONAL RESEARCH COUNCIL

# NATIONAL RESEARCH COUNCIL
## DIVISION OF PHYSICAL SCIENCES

## COMMITTEE ON MATHEMATICAL TABLES AND OTHER AIDS TO COMPUTATION

*Professor R. C. ARCHIBALD, *chairman*, Brown University, Providence 12, Rhode Island (R.C.A.)
*Professor S. H. CALDWELL, Department of Electrical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts (S.H.C.)
*Doctor L. J. COMRIE, Scientific Computing Service, Ltd., 23 Bedford Square, London, W.C. 1, England (L.J.C.)
*Professor H. T. DAVIS, Department of Mathematics, Northwestern University, Evanston, Illinois (H.T.D.)
*Doctor W. J. ECKERT, Watson Scientific Computing Laboratory, 612 West 116th St., New York City 27 (W.J.E.)
*Mister J. S. ELSTON, The Travelers, Hartford, Connecticut (J.S.E.)
*Professor D. H. LEHMER, Ballistic Research Laboratory, Aberdeen Proving Ground, Md. (D.H.L.)
*Professor S. S. WILKS, Department of Mathematics, Princeton University, Princeton, New Jersey (S.S.W.)
Professor H. H. AIKEN, Cruft Laboratory, Harvard University, Cambridge 38, Mass.
Professor W. G. COCHRAN, Iowa State College of Agriculture and Mechanic Arts, Ames, Iowa
Professor C. EISENHART, Agronomy Building, University of Wisconsin, Madison, Wisconsin
Professor J. D. ELDER, Department of Mathematics, University of Michigan, Ann Arbor, Michigan
Doctor L. GOLDBERG, McMath-Hulbert Observatory, Route 4, Pontiac, Michigan
Professor P. W. KETCHUM, Department of Mathematics, University of Illinois, Urbana, Illinois
Miss C. M. KRAMPE, U. S. Naval Observatory, Washington
Professor T. KUBOTA, Tôhoku University, Sendai, Japan, Representative of the National Research Council of Japan
Doctor A. N. LOWAN, 312 Schenectady Avenue, Brooklyn 13, New York
Doctor J. C. P. MILLER, (Department of Applied Mathematics, University of Liverpool), 18 Garthdale Road, Liverpool 18, England
Doctor G. R. STIBITZ, University of Vermont, Burlington, Vermont
Mister J. S. THOMPSON, Mutual Benefit Life Insurance Company, Newark, New Jersey
Professor I. A. TRAVIS, [Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, Pennsylvania] Lieutenant Commander Travis, Bureau of Ordnance, Navy Department, Washington, D. C.
Mister W. R. WILLIAMSON, Federal Security Agency, Social Security Board, Washington
Mister J. R. WOMERSLEY, National Physical Laboratory, Teddington, Middlesex, England

* Member of the Executive Committee.


---


Published quarterly in January, April, July and October by the National Research Council, Prince and Lemon Sts., Lancaster, Pa., and Washington, D. C.

All contributions intended for publication in *Mathematical Tables and Other Aids to Computation*, and all Books for review, should be addressed to Professor R. C. ARCHIBALD, Brown University, Providence, R. I.

Entered as second-class matter July 29, 1943, at the post office at Lancaster, Pennsylvania, under the Act of August 24, 1912.

IN THE DIFFERENTIAL ANALYZER ROOM. 29 OCTOBER 1945
H. B. PHILLIPS, V. BUSH, H. L. HAZEN, S. H. CALDWELL

# Conference on Advanced Computation Techniques

During the recent war there was tremendous development of certain types of computing devices. The big Harvard-IBM machine indicates the way in which a whole battery of IBM equipment can be brought under automatic electric control. Electronic devices have been constructed, or are under development, which promise astronomical speeds for numerical processes. At the Bell Laboratories there has been important development of automatic relay computing. Thus, with war's end these and other similar developments suggest that there will soon be available mechanical and electrical computing equipment which, in terms of speed and flexibility, will completely outdistance anything thought of before.

With this in mind the desirability of a Conference of those who, in this country and in England, have been recently directly and indirectly connected with the output of large computing machines became evident, and the National Research Council Committee on Mathematical Tables and Other Aids to Computation took steps for its organization. Most of the responsibility of this organization was placed in the hands of the Committee's Subcommittee Z (On Calculating Machines and Mechanical Computation) which then[1] had the following membership:

| | |
|---|---|
| L. J. Comrie, *Chairman* | J. C. P. Miller |
| S. H. Caldwell, *Vice-Chairman* | G. R. Stibitz |
| H. H. Aiken | I. A. Travis |
| D. H. Lehmer | J. R. Womersley. |

Since the chairman had not yet arrived in this country, the vice-chairman carried out most of the detailed arrangements.

The Massachusetts Institute of Technology not only offered its facilities for such a gathering but also suggested that it should coincide with the first public demonstration of its new Differential Analyzer. The dates for the gathering were fixed as October 29, 30, 31, 1945. There were the following 84 guests who accepted our invitation:

| *Name* | *Representing* |
|---|---|
| Abramowitz, Milton | Math. Tables Project, 150 Nassau St., New York City |
| Aiken, H. H. (Commander) | Computation Project, Cruft Laboratory, Harvard University |
| Alt, Franz (Lt.) | Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland |
| Archibald, R. C. (Prof.) | Nat. Res. Council Comm. (Brown University) |
| Arnold, H. A. (Lt. Commander) | Computation Project, Cruft Laboratory |
| Bennot, Maude (Miss) | 25 Ridgeway Avenue, Needham, Mass. |
| Bergman, Stefan (Dr.) | Brown University |
| Berkeley, E. C. (Lt.) | Computation Project, Cruft Laboratory |
| Block, R. M. | Computation Project, Cruft Laboratory |
| Bode, H. W. (Dr.) | Bell Telephone Laboratories, New York City |
| Brainerd, J. G. (Prof.) | University of Pennsylvania |
| Brendle, Ruth (Miss) | Computation Project, Cruft Laboratory |

| *Name* | *Representing* |
|---|---|
| Bryant, R. C. (Lt.) | Office of Research & Inventions, 1 Park Avenue, New York City |
| Bush, Vannevar (Dr.) | Carnegie Institution, Washington, D. C. |
| Caldwell, S. H. (Prof.) | Nat. Res. Council Comm. (Mass. Inst. of Technology) |
| Campbell, R. V. D. (Ensign) | Computation Project, Cruft Laboratory |
| Carter, G. K. | General Electric Co., Schenectady, N. Y. |
| Colbath, D. L. | General Electric Co. |
| Comrie, L. J. (Dr.) | Nat. Res. Council Comm. (London, England) |
| Concordia, Charles | General Electric Co. |
| Crawford, P. O. jr. | Office of Research & Inventions |
| Cunningham, L. E. (Dr.) | Ballistic Research Laboratory |
| Curry, H. B. (Dr.) | Ballistic Research Laboratory |
| Dederick, L. S. (Dr.) | Ballistic Research Laboratory |
| Desch, J. R. | National Cash Register Co., Dayton, Ohio |
| Douglas, E. M. | International Business Machines Corp., 590 Madison Ave., New York City |
| Dwight, H. B. (Prof.) | Mass. Inst. of Technology |
| Eckert, J. P. jr. | University of Pennsylvania |
| Forrester, J. W. | Mass. Inst. of Technology |
| Goldstine, H. H. (Capt.) | Ballistic Research Laboratory |
| Gore, O. L. (Lt.) | Office of Research & Inventions |
| Hamilton, F. E. | International Business Machines Corp. |
| Haskins, Robert (Lt.) | Office of Research & Inventions |
| Hazen, H. L. (Prof.) | Mass. Inst. of Technology |
| Henry, W. E. | Mass. Inst. of Technology |
| Hopper, Grace (Lt.) | Computation Project, Cruft Laboratory |
| Hrones, J. A. (Prof.) | Mass. Inst. of Technology |
| Jordan, W. B. | General Electric Co. |
| Kelly, T. S. (Prof.) | Institute Math. Statistics (Harvard University) |
| Kopal, Zdenek (Dr.) | Mass. Inst. of Technology |
| Kuehni, H. P. | General Electric Company |
| Lehmer, D. H. (Prof.) | Nat. Res. Council Comm. (Ballistic Research Laboratory) |
| Löf, J. L. C. | Mass. Inst. of Technology |
| Lowan, A. N. (Dr.) | Math. Tables Project |
| Macleish, K. G. | Eastman Kodak Co., Rochester, N. Y. |
| McPherson, J. C. | International Business Machines Corp. |
| Maginniss, F. J. | General Electric Co. |
| Mauchly, J. W. (Prof.) | University of Pennsylvania |
| Mayall, Margaret (Mrs.) | Harvard Observatory, Harvard University |
| Mork, R. G. | Office of Research & Inventions |
| Mumma, R. E. | National Cash Register Co. |
| Myers, J. E. (Ensign) | Naval Research Laboratory, Anacostia, Md. |
| von Neumann, John (Prof.) | Institute Math. Statistics (Institute for Advanced Study) |
| Nyquist, Harry (Dr.) | Bell Telephone Laboratories |
| O'Neal, R. D. | Eastman Kodak Company, Rochester, N. Y. |
| Peterson, H. A. | General Electric Co. |
| Philbrick, G. A. | Foxboro Instrument Company, Foxboro, Mass. |
| Pitts, W. H. | Mass. Inst. of Technology |
| Rankin, A. W. | General Electric Co. |
| Rees, Mina S. (Dr.) | Applied Mathematics Panel (350 Fifth Avenue, New York City) |
| Roosevelt, C. V. S. (Lt.) | Office of Research & Inventions |

| *Name* | *Representing* |
|---|---|
| Sandomire, Marion M. (Mrs.) | Bureau of Ships, Washington, D. C. |
| Sarahan, B. L. (Lt.) | Naval Research Laboratory |
| Schultz, O. T. | Bell Telephone Laboratories |
| Shannon, C. E. (Dr.) | Bell Telephone Laboratories |
| Sternberg, Sidney (Lt.) | Office of Research & Inventions |
| Stibitz, G. R. (Dr.) | University of Vermont, Burlington, Vt. |
| Svihel, B. T. | Mass. Inst. of Technology |
| Taylor, Richard (Prof.) | Mass. Inst. of Technology |
| Thomas, G. B. jr. (Prof.) | Mass. Inst. of Technology |
| Tiffany, O. L. | University of Michigan |
| Tukey, J. W. (Prof.) | Princeton University |
| Tyler, A. W. | Eastman Kodak Co., Rochester, N. Y. |
| Verzuh, F. M. | Mass. Inst. of Technology |
| Wakelin, J. W. jr. (Lt.) | Office of Research & Inventions |
| Wallman, Henry (Dr.) | Mass. Inst. of Technology |
| Warren, S. R. jr. (Prof.) | University of Pennsylvania |
| Watson, F. G. (Lt. Commander) | Hydrographic Office, Navy Dept., Washington, D. C. |
| Wicklund, H. P. (Lt.) | Office of Research & Inventions |
| Wiener, Norbert (Prof.) | Mass. Inst. of Technology |
| Williams, H. M. | National Cash Register Co. |
| Williams, S. B. | Bell Telephone Laboratories |
| Wood, H. A. (Dr.) | Chance-Vought Aircraft Div. of the United Aircraft Corp., Stamford, Conn. |
| Wundheiler, A. W. (Dr.) | Bureau of Ordnance |

Most of these delegates arrived on October 29 and attended demonstrations of the working of the new Differential Analyzer, of which there is a detailed illustrated description by VANNEVAR BUSH and S. H. CALDWELL,[2] in Franklin Institute, *J.*, v. 240, Oct. 1945, p. 255–326. Our frontispiece reproduces a photograph taken at the opening session of the Differential Analyzer demonstration. Grouped about the Main Control Panel are (3) Mr. Hazen, head of the Department of Electrical Engineering at M.I.T., (2 & 4) Messrs. Bush & Caldwell, who were initially responsible for developing the Analyzer, and (1) Mr. Phillips, head of the Department of Mathematics at M.I.T. In the evening the Institute invited the Delegates to a delightful dinner at the Engineers Club on Commonwealth Ave. The addresses of JAMES R. KILLIAN, JR., executive vice-president of the Institute, *presiding officer*, GEORGE R. HARRISON, dean of science at the Institute, LESLIE J. COMRIE, chairman of Section Z, and VANNEVAR BUSH, director of the Carnegie Institution of Washington, were uniformly of exceptionally happy content.

The Program carried out on Tuesday, Wednesday, October 30, 31, with L. J. COMRIE and S. H. CALDWELL presiding, was as follows:

*Tuesday Morning*, M.I.T.

Address of Welcome to the Institute by President K. T. COMPTON.

Introductory Remarks by VANNEVAR BUSH.

"Some Industrial Applications of Machine Computing Methods" by F. J. MAGINNISS.

"Numerical Computation by the Use of Telephone Relays" by S. B. WILLIAMS.

*Tuesday Afternoon*, M.I.T.

"Some Principles of Electronic High Speed Computing" by J. G. BRAINERD and J. P. ECKERT, JR.

"A General Summary and a Look at the Future" by J. VON NEUMANN.

*Tuesday Evening*, Cruft Laboratory, Harvard University

"The Automatic Sequence Calculator"[2] by H. H. AIKEN, and members of his staff.

*Wednesday Morning*, M.I.T.

"Simulative Techniques as an Aid to Analysis" by G. A. PHILBRICK.

"Application of Punched-Card Methods to Scientific Computation" by W. J. ECKERT, presented by J. C. MCPHERSON.

"The M.I.T. Center-of-Analysis Research Program" by S. H. CALDWELL.

*Wednesday Afternoon*, M.I.T.

"The Scientific Application of Commercial Calculating Machines" by L. J. COMRIE.

The Conference was most notably successful, and one heard on every side expressions of the hope that such a Conference might become an annual event.

The Committee on Mathematical Tables and Other Aids to Computation is indeed greatly indebted to officers of the administration and members of the staff of the Massachusetts Institute of Technology who did everything in their power to provide a setting for securing fruitful results from such an assembly of experts.

R. C. A.

[1] Beginning with Nov. 1, 1945, S. H. C. became *chairman*, and L. J. C. *vice-chairman*.
[2] See MAC 19.
[3] For illustrated notes on this Calculator see MAC 20.

## RECENT MATHEMATICAL TABLES

**275[A, E, F].**—PEDER PEDERSON, (a) *Über die numerische Berechnung der Kettenbrüche nebst einer Berechnung der Grundzahl der natürlichen Logarithmen*, 36 p., 1940; (b) *Berechnung der Grundzahl e der natürlichen Logarithmen mit 606 Dezimalen*, 17 p., 1942; (c) *Fortsetzung der Berechnung der Grundzahl e der natürlichen Logarithmen bis zur 808. Dezimalstelle*, 21 p., 1944. Denmark, Geodætisk Institut, *Meddelelse*, nos. 14, 16, and 17 respectively. 14.3×22 cm.

These three papers are accounts of the author's calculations of the fundamental constant $e$. Values of $e$ are given to 404D in (a), to 606D in (b) and to 808D in (c). The method of calculation was based on the continued fraction

$$\frac{e-1}{2} = \frac{1}{1} + \frac{1}{6} + \frac{1}{10} + \frac{1}{14} + \cdots$$

which is due to EULER,[1] but is applicable to any regular continued fraction. The general idea is set forth in (a) and may be roughly described as follows. Let $x$ be the exact value of a continued fraction and let $x_n$ be a given approximation to $x$ which is correct to $n$ decimals. Let also $A_k/B_k$ be any convergent of the continued fraction in which $k$ is large enough so that the value of this convergent would give $x$ to more than $n$ decimals. This means that, roughly speaking, $B_k$ has about $n/2$ digits. Then $x_kB_k$ will be a decimal whose first $n/2$ significant figures are the digits of $A_k$ and whose next $n/2$ digits are zeros. Therefore if we compute denominators $B_k$ only, and compute only the digits of the product $x_kB_k$ which lie near the $n$th significant figure, the appearance of zeros (or nines) at this place serves as a check on the digits of $x_n$. Moreover, the method may be used to extend the value of $x$ beyond the $n$th decimal place by simply filling in those digits which will produce more zeros until the accuracy implicit in a particular $B_k$ is exceeded.

The author started with SHANKS' 137D value of $e$ verified by GLAISHER,[2] and went on to check the value of Shanks[3] (205D, last 18 wrong) and of BOORMAN[4] (346D, last 123 wrong). The 223 correct decimals of Boorman was then the basis of an extension to

404D, and then to 606D. At this point the author became aware of a 707D value given in 1926 by the reviewer,[5] which he verified and used to extend the value of $e$ to the impressive result of 808D. The suggested method of checking and extending the value of a continued fraction, given in 1926, is thus independently discovered and greatly elaborated by the author.

The determination of the digits of $e$, one at a time, seems to suggest that the author used no calculating machine to perform this immense calculation.

Besides the values of $e$ mentioned above the author gives all the denominators of the convergents up to $B_{170}$, a 408 digit integer. These interesting numbers begin with $B_1 = 1$, $B_2 = 7$, $B_3 = 71$, $B_4 = 1001$ and were found by the recurrence formula

$$B_{k+1} = (4k + 2)B_k + B_{k-1}.$$

They have interesting divisibility properties discovered by D. N. LEHMER[6] and further studied by the reviewer.[7] The author gives no description of this part of the work and of the checks which are available to insure the accuracy of this, the most hazardous part of the calculation.

Since the periodical in which this value of $e$ to 808D appears is highly inaccessible in this country, it seemed advisable to reproduce it below.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $e$ = 2.71828 | 18284 | 59045 | 23536 | 02874 | 71352 | 66249 | 77572 | 47093 | 69995 |
| 95749 | 66967 | 62772 | 40766 | 30353 | 54759 | 45713 | 82178 | 52516 | 64274 |
| 27466 | 39193 | 20030 | 59921 | 81741 | 35966 | 29043 | 57290 | 03342 | 95260 |
| 59563 | 07381 | 32328 | 62794 | 34907 | 63233 | 82988 | 07531 | 95251 | 01901 |
| 15738 | 34187 | 93070 | 21540 | 89149 | 93488 | 41675 | 09244 | 76146 | 06680 |
| 82264 | 80016 | 84774 | 11853 | 74234 | 54424 | 37107 | 53907 | 77449 | 92069 |
| 55170 | 27618 | 38606 | 26133 | 13845 | 83000 | 75204 | 49338 | 26560 | 29760 |
| 67371 | 13200 | 70932 | 87091 | 27443 | 74704 | 72306 | 96977 | 20931 | 01416 |
| 92836 | 81902 | 55151 | 08657 | 46377 | 21112 | 52389 | 78442 | 50569 | 53696 |
| 77078 | 54499 | 69967 | 94686 | 44549 | 05987 | 93163 | 68892 | 30098 | 79312 |
| 77361 | 78215 | 42499 | 92295 | 76351 | 48220 | 82698 | 95193 | 66803 | 31825 |
| 28869 | 39849 | 64651 | 05820 | 93923 | 98294 | 88793 | 32036 | 25094 | 43117 |
| 30123 | 81970 | 68416 | 14039 | 70198 | 37679 | 32068 | 32823 | 76464 | 80429 |
| 53118 | 02328 | 78250 | 98194 | 55815 | 30175 | 67173 | 61332 | 06981 | 12509 |
| 96181 | 88159 | 30416 | 90351 | 59888 | 85193 | 45807 | 27386 | 67385 | 89422 |
| 87922 | 84998 | 92086 | 80582 | 57492 | 79610 | 48419 | 84443 | 63463 | 24496 |
| 84875 | 602... | | | | | | | | |

D. H. L.

[1] L. EULER, *Introductio in Analysin Infinitorum*, v. 1, Lausanne, 1748, p. 319; *Opera Omnia*, s. 1, v. 8, Leipzig and Berlin, 1922, p. 388.

[2] J. W. L. GLAISHER, B.A.A.S., *Report*, 1871, p. 16–18 (Sect. trans.), also *Report*, 1873, p. 128, l. 12.

[3] W. SHANKS, Royal So. London, *Proc.*, v. 6, 1854, p. 397.

[4] J. M. BOORMAN, *Math. Mag.*, v. 1, 1884, p. 204. Another calculation was made by Fr. Tichâneck in 1893 to 225D (correct to within a unit in the last place), *Jahrbuch ü. d. Fortschritte d. Mathem.*, v. 25, p. 736; there is here a misprint in the 43rd decimal place, *for* 6 *read* 0.

[5] D. H. LEHMER, *Amer. J. Math.*, v. 48, 1926, p. 139–143.

[6] D. N. LEHMER, *Amer. J. Math.*, v. 40, 1918, p. 375–390. Also Nat. Acad. Sci., *Proc.*, v. 4, 1918, p. 214–218.

[7] D. H. LEHMER, *Annals Math.*, v. 23, 1932, p. 143–150.

EDITORIAL NOTE: For various values of $e^n$, see *MTAC*, v. 1, p. 54–55.

**276[B, K, M].**—A. N. KOLMOGOROV, (a) "Chislo popadaniĭ pri neskol'kikh vystrelakh i obshchie princ̆sipy oc̆senki effektivnosti sistemy strel'by" (The number of hits in several rounds, and the general principles of estimating effectiveness of a system of firing); (b) "Iskusstvennoe rasseivanie v sluchae porazheniiâ odnim popadaniem i rasseivaniiâ v

odnom izmerenii" (Intentional dispersions in case one hit is fatal and dispersions in one dimension), Akademiîa Nauk SSSR, Leningrad, Matematicheskiĭ Institut imeni V. A. Steklova, *Trudy*, v. 12, June 194-, p. 7–45, 95–106.

This issue of the *Trudy* is devoted to a collection of articles on the theory of gunnery, edited by Kolmogorov. There are four tables (p. 96–106). **Table 1** gives values of

$$H(m, a) = 1 - e^{-a}\left(1 + \frac{a}{1!} + \frac{a^2}{2!} + \cdots + \frac{a^{m-1}}{(m-1)!}\right),$$

for $a = [.1(.1)15; 5D]$, $m = 1(1)11$.

For the same range **T. 2** gives the values of

$$\nabla^2 H(m, a) = H(m, a) - 2H(m-1, a) + H(m-2, a).$$

In **T. 3**, values are given for

$$\tau(u) = \frac{1}{2u}\int_{-\infty}^{+\infty} \ln\{1 - \tfrac{1}{2}[\Phi(z+u) - \Phi(z-u)]\}dz$$

for $u = [0(.05)1(1)5; 3D]$, with $\Delta$; $\Phi(z) = \dfrac{2\rho}{\sqrt{\pi}}\int_0^z e^{\rho^2 z^2}du$, $\rho = .476936\ldots$

**T. 4** gives the values of $P(N) = \Phi(s_0) - 2s_0\phi(s_0)$, $\phi(z) = e^{\rho^2 z^2}/\pi^{\frac{1}{2}}$, $s_0(N) = \rho^{-1}(\frac{3}{4}\pi^{\frac{1}{2}})N^{\frac{1}{3}}$, and $d^2(N) = \frac{1}{3}s_0^3$, for $N = [0(.1)1(.2)5(.5)10; 3S]$ with $\Delta$.

On p. 104–106 are graphs of (a) $H(m, a)$, $m = 1(1)10$, $a = 0(1)15$; (b) $R(k, c, n)$, $k = \frac{1}{2}$, $c = 4/15$, $n = 2(2)6$, $12(6)24$, $0 < a < 4.8$, the values of $R$ being given in a table on p. 33 for given values of $n$ and $a$; (c) $R$, $k = \frac{1}{2}$, $c = 4/15$, $n = 3$, $6(6)24$, $0 < a < 2.4$, the values of $R$ being given on p. 34 for values of $n$ and $a$.

R. C. A.

**277[D, J].**—Great Britain, H.M. Nautical Almanac Office, *Tables for the Summation of Trigonometrical Series*. Department of Scientific Research and Experiment, Admiralty Computing Service, November, 1945. No. SRE/ACS. 93. 24 leaves mimeographed on one side of the paper. 20.2×32.9 cm. This publication is available only to certain Government agencies and activities.

These tables give the values of $\sin nx$ and $\cos nx$ for $n = 1(1)10$, $x = 0(1°)180°$, and also for radian arguments $x = 0(.01)3.14$. The values corresponding to $n = 1$, 2, 3, and 4 are given to 5D; those corresponding to $n = 5$, 6, and 7 to 4D; and those corresponding to $n = 8$, 9, and 10 to 3D.

These tables were compiled for use in the summation of trigonometric series of the form $\Sigma(A_n \cos nx + B_n \sin nx)$ when the convergence is fast enough for the sum of the first ten terms to give adequate accuracy. In conjunction with a table of powers, the tables may also be used to convert polar into cartesian coordinates.

Arnold N. Lowan

Editorial Note: For radian argument the above-mentioned table ranges, at varying intervals, from 0 to 31.4 radians, 5–3D. The NYMTP *Tables of Sines and Cosines for Radian Arguments*, 1940, give, among other tables, $\sin x$ and $\cos x$, for $x = [0(.001)25; 8D]$, and $x = [0(1)100; 8D]$.

**278[E, M].**—Louis Dunoyer, "Calculs sur l'émission en bout des tubes lumineux et notamment des tubes électro- ou photoluminescents," *Revue d'Optique*, v. 18, 1939, p. 235–241. 16×24.2 cm.

There are several tables and graphs. The tables are as follows:

(a) $y = e^{-a/\sin\gamma}\sin\gamma$, for $a = .5$, and $a/\sin\gamma = [.5(.01).53, .55(.5).75, .85, 1(.25)1.5(.5)3(1)6; 5D]$.

(b) $H(m) = \int_u^{\frac{1}{2}\pi} e^{-a/\sin\gamma} \sin\gamma d\gamma$, for $u = [0°, 2°(4°)90°; 5D]$, and $a = .5$, $m = \cot u$.

(c) $H(m) = \int_u^{\frac{1}{2}\pi} e^{-a/\sin\gamma} \sin\gamma d\gamma$, for $a = 0, .2, .5, 1, 2$, and $m = [0, .01(.02).99, 1.05(.1)1.95,$ also $.1(.2)8.5, 9.5; 4D]$.

(d) $\int_0^M [H(m) + H(M-m)]\,\Lambda(m)dm$, $\Lambda(m) = \int_0^\infty e^{-kz/\cos\alpha} \sin\alpha d\alpha$,

$H(M-m) = \int_{u'}^{\frac{1}{2}\pi} e^{-a/\sin\gamma} \sin\gamma d\gamma$, $\cot u' = M - m$, for

$M = [0(.1)1(.2)4(1)10, \infty; 4D]$ and $a = k = 0, .2, .5, 1, 2$.

There are graphs of $H(m)$, $a = k = 0, .2, .5, 1, 2$, and $0 < m \leqslant 5$; of $Hm + H(M-m)$ for $M = .2(.2)1(.5)2(1)5$, $0 < m \leqslant 5$; and of $\int_0^M [H(m) + H(M-m)]\,\Lambda(m)dm$ for $a = k = 0, .2, .5, 1, 2$, and $0 < m \leqslant 4$.

R. C. A. & S. A. JOFFE

**279[F].**—N. G. W. H. BEEGER, "Extension of the table of least exponents $\xi$ for which $2^\xi \equiv 1 \pmod p$)." *Nieuw Archief v. Wiskunde*, s. 2, v. 20, 1940, p. 307–8. 15.7×23.6 cm.

The number $\xi$ referred to in the title is a function of $p$ familiar to number theorists and usually called the exponent to which 2 belongs (mod $p$). It is known that $\xi$ is some divisor of $p - 1$. The other factor $\nu = (p-1)/\xi$ has been called by A. J. C. CUNNINGHAM the residue index of 2 (mod $p$) and this function, because it is nearly always much smaller than $\nu$, is usually tabulated in lieu of $\xi$. The present note contains a one-page table giving $\nu$ for each prime $p$ between 300000 and 310000. As a matter of fact those cases for which $\nu = 1$ and 2 are omitted in the table to save space. A missing $p$ has $\nu = 1$ or 2 according as $(p^2 - 1)/8$ is odd or even. This table is an extension of the large table of $\nu$ for $p < 300000$ published by M. KRAITCHIK, in his *Recherches sur la Théorie des Nombres*, v. 1, Paris, 1924, p. 131–191. Tables of $\nu$, like so many other difficult number-theoretic functions, must be constructed one entry at a time, and each entry must stand on its own merits, without hope of a smoothness check. Thus a small table like the present one represents a considerable amount of careful computation.

The writer has in his possession a photostat copy of a manuscript containing an extension by Beeger of the table here reviewed; it covers the primes between 310000 and 320000.

Arithmeticians will be glad to learn that Mr. Beeger not only survived the war but was able to continue his researches in the theory of numbers.

D. H. L.

**280[F].**—A. GLODEN, "Table des solutions de la congruence $X^4+1 \equiv 0 \pmod p$ pour $2 \cdot 10^5 < p < 3 \cdot 10^5$," *Mathematica* (Rumania), v. 21, 1945, p. 45–65.

This table is an extension of two previous tables:

A. J. C. CUNNINGHAM, *Binomial Factorisations*, v. 1, London, 1923, p. 1–95, and v. 2, 1924, p. 1–104, for $p < 10^5$; and S. HOPPENOT, "Table des solutions de la congruence $x^4 \equiv -1 \pmod N$ pour $100000 < N < 200000$," Brussels, Librairie du *Sphinx*, 1935, 18 p.; see *MTAC*, RMT 48, v. 1, p. 6.

In order that the congruence

$$X^4 + 1 \equiv 0 \pmod p \tag{1}$$

have a solution it is necessary and sufficient that the prime $p$ be of the form $8n + 1$. Accordingly, the table is concerned with such primes only. These primes $p$ are capable of being represented in essentially one way by the quadratic forms

$$p = x^2 + y^2 = 2z^2 + t^2.$$

If one solves the linear congruences

$$x \equiv uy \pmod{p} \quad \text{and} \quad z \equiv vt \pmod{p}$$

for $u$ and $v$, then the four solutions of the quartic congruence (1) are given by

$$X = \pm u(v \pm 1).$$

The table gives for each $p = 8n + 1$ between 200000 and 300000 two values of $X$ which are less than $p/2$; the other two values of $X$ are of course negatives of these modulo $p$. There is one curious exception at $p = 200569$ where, for some reason, the second solution is given as 100317 instead of 100252. Since the product of all the roots of (1) is clearly congruent to unity modulo $p$, the product of any two roots is congruent to $\pm 1$ modulo $p$. This fact was used as a check of the table.

At the end of the table is given a list of 25 numbers of the form $x^4 + 1$ and two of the form $x^8 + 1$ with their complete decompositions into primes made possible by this table. For example

$$1620^4 + 1 = 289889 \cdot 23759009.$$

The first factor follows from the entry

$$p = 289889, \quad X_1 = 1620, \quad X_2 = 15926$$

of the main table. The second factor is clearly a prime having no factors $< 300000$. A more extensive factor table of 54 entries (including only 8 of the above 27) based presumably upon a corresponding table of solutions of (1) up to $p < 500000$, appeared in *Mathesis*, v. 55, 1945, p. 81–82.

It is perhaps worth pointing out that the main table is useful as a list of primes of the form $8n + 1$ even if one is not concerned with the quartic congruence (1).

D. H. L.

**281[F].**—V. THÉBAULT, *Les Récréations Mathématiques* (*Parmi les nombres curieux*), Supplément à *Mathesis*, Gembloux, p. 1–80, v. 54, no. 10, Aug. 1943; p. 81–119, no. 11, Feb. 1945. 16.3×25.2 cm.

Here are numerous independent tables, not all published for the first time, including the following:

**1.** With the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken once, form two numbers of three digits whose sum is equal to numbers of (a) three digits chosen once; (b) four digits chosen once. On p. 8, for (a) a table of 21 solutions is given, and for (b) 48. For example, $286 + 173 = 459$; $789 + 246 = 1035$. For (a) see *L'Éducation Math.*, v. 23, 1920, p. 34–35.

**2.** With the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, taken once, form an integer which is a square. On p. 11, 30 solutions are given from $11826^2 = 139854276$, to $30384^2 = 923187456$; 29 of these were published earlier in *Mathesis*, v. 51, 1937, p. 279–281. On p. 12 are 87 squares formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken once, from $32043^2 = 1026753849$, to $99066^2 = 9814072356$.

**3.** P. 62–68 gives all 1-digit, 2-digit, 3-digit, and 4-digit square endings for the base 10, together with an inverse table showing the forms of those numbers whose squares end in a specified 1-digit, 2-digit or 3-digit number. The author has published this before in *Mathesis*, v. 48 Suppl. Oct. 1934, 22 p. A slightly more complete table is due to SCHADY, *J. f. d. reine u. angew. Math.*, v. 84, 1877, p. 86–88.

**4.** "Table des carrés des nombres entiers de 1 à 1000 dans les systèmes de numération de bases $B = 2, 3, 4, 5, 6, 7, 8, 9, 11, 12$," p. 89–119.

These four tables are of interest to "digitologers," a sect of which the author is a leading exponent. Table **3**, however, is of practical use in the rapid identification of non-squares.

D. H. L.

**282[I].**—HERBERT E. SALZER, "Formulas for direct and inverse interpolation of a complex function tabulated along equidistant circular arcs," *J. Math. Phys.*, v. 24, Nov. 1945, p. 141–143. 17.4×26.4 cm.

The purpose of this paper is to facilitate direct as well as inverse interpolation of an analytic function $f(z)$ whose numerical values are known in a sequence of points $z_\nu$ which are equally spaced along a circle in the complex plane. In terms of the new variable $P = (z - z_0)/(z_1 - z_0)$ the problem reduces to the case when the functional values are given at the points $P_\nu$ defined by

$$P_0 = 0,\ P_\nu = 1 + \omega + \omega^2 + \cdots + \omega^{\nu-1}\ (\nu \geqq 1)$$
$$P_\nu = -(\omega^{-1} + \omega^{-2} + \cdots + \omega^{\nu})\ (\nu \leqq -1),\ (\omega = e^{i\theta}).$$

Let

$$F(P) = \sum_{\nu=\mu-n+1}^{\mu} L_\nu^{(n)}(P) F(P_\nu),\ (\mu = [n/2]),$$

be the $n$-point Lagrange-Hermite interpolation formula based on our points $P_\nu$. Setting $P^r = p_r + iq_r$ $(r = 0, 1, 2, \cdots)$, the paper furnishes explicit expressions, in terms of $p_r$, $q_r$ and the angle $\theta$, of the real and imaginary parts of the coefficients $L_\nu^{(n)}(P)$, for $n = 3, 4$, and 5. It is also shown how these expressions may be used to apply also in the present situation the author's recent formula for inverse interpolation (see "A new formula for inverse interpolation," Amer. Math. So., *Bull.*, v. 50, 1944, p. 513–516).

I. J. SCHOENBERG

Univ. of Pennsylvania

**283[I].**—HERBERT E. SALZER, "Inverse interpolation for eight-, nine-, ten-, and eleven-point direct interpolation," *J. Math. Phys.*, v. 24, May 1945, p. 106–108. 17.4×26.4 cm.

This paper extends a formula which was first published by the author as "A new formula for inverse interpolation," Amer. Math. So., *Bull.*, v. 50, 1944, p. 513–516. The expression in question is the inversion of the Lagrange interpolation formula.

If we write $f_x = f(a_0 + xh)$, where $h$ is the tabular interval, and if $p$ is desired when we are given $f_p$, then $p$ can be expressed in terms of the tabulated values, $f_0, f_1, f_2, \cdots f_{-1}, f_{-2}$, etc. by means of the formula:

$$p = \sum_{m=1}^{k-1} \frac{(f_p - f_0)^m}{m!} \left[ \frac{d^{m-1}}{da^{m-1}} \left( 1 \Big/ \sum_{i=-\frac{1}{2}(n-1)}^{\frac{1}{2}n} \frac{L_i^{(n)}(a)}{a} (f_i - f_0) \right)^m \right]_{a=0} + R_k,$$

in which $L_i^{(n)}(a)$ is a polynomial of $(n-1)$ degree in $a$.

The author reduced this expression to the form

$$p = r - r^2 s + r^3(2s^2 - t) + r^4(-5s^3 + 5st - u) + \cdots$$

where the quantities $r$, $s$, $t$, $u$, etc. are expressed in terms of the tabulated values of the function.

In the present paper the explicit values of these variables are given for the case where eight-, nine-, ten-, or eleven-point direct interpolation is used.

It is difficult to imagine when an equation of this extent may be required in practical work, but the author remarks: "its full use can provide unusual accuracy in solving equations (both real and complex) up to the tenth degree when the values of the polynomials are tabulated near the root at equal intervals."

H. T. D.

**284[I].**—HERBERT E. SALZER, "Table of coefficients for double quadrature without differences, for integrating second order differential equations," *J. Math. Phys.*, v. 24, Nov. 1945, p. 135–140. 17.4×26.4 cm.

A function $f(x)$ may be expressed in terms of its second derivative as

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} f''(x_2)dx_2. \tag{1}$$

If numerical values of $f''(x)$ are given in the $n$ points

$$x_p = x + ph \quad (p = \mu - n + 1, \cdots, \mu, \text{ where } \mu = [n/2]),$$

we may approximate $f''(x)$ by the $n$-point Lagrange formula. A double integration of the interpolating polynomial leads from (1) to the approximation formula

$$f(x_0 + ph) = f(x_0) + phf'(x_0) + h^2 \sum_{i=\mu-n+1}^{\mu} C_{i,n}(p)f''(x_0 + ih) + R_p, \tag{2}$$

where $C_{i,n}(p)$ are polynomials in $p$ of degree $n + 1$, given by

$$C_{i,n}(p) = (-1)^{i+\mu} \int_0^p dp_1 \int_0^{p_1} \left\{ \prod_{\lambda=\mu-n+1}^{\mu}{}' (p_2 - \lambda) \right\} dp_2 / \{(i + n - \mu - 1)!\,(\mu - i)!\}, \tag{3}$$

the prime on the product sign indicating that the factor $p_2 - i$ is omitted. The paper furnishes for $n = 3(1)11$, the numerical (rational) values of the coefficients (3) for $p = (\mu - n + 1)(1)\mu$, reduced to their least common denominator.

The author discusses briefly the application of the formula (2) to the numerical integration of a differential equation of the type

$$y'' + \phi(x, y) = 0,$$

to which any second order linear differential equation may be reduced by a proper change of unknown function. The solution once "started," the familiar procedure of the progressive refinement of a first approximation to a new value of $y$ applies, by working back and forth between the columns of values of $y$ and $y''$. At each step, however, the value of $hf'(x_0)$ must also be determined by (2) for any single value of $p$ ($\neq 0$) for which $f(x_0 + ph)$ is already known. Thus a column of values of $hy'$ is also determined. The author concludes by pointing out further useful applications to the integration of partial differential equations of the type

$$u_{xx} = u_t + \psi(u, x, t).$$

I. J. SCHOENBERG

**285[K].**—ØYVIND BURRAU, "Middelfejlen som Usikkerhedsmaal," *Matematisk Tidsskrift*, 1943B, p. 9–16. 15.1×24 cm.

Suppose $x_1, x_2, \cdots, x_n$ are values of $x$ in a sample of $n$ items from a normal population with mean $a$ and standard deviation $\sigma$. W. S. GOSSETT, under the pseudonym "Student,"[1] in 1908 conjectured that the sampling distribution of

$$z = \frac{\bar{x} - a}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2}}$$

was given by[2]

$$f(z)dz = C(1 + z^2)^{-\frac{1}{2}n} dz \tag{1}$$

where $C$ is a constant, such that $\int_{-\infty}^{\infty} f(z)dz = 1$. This conjecture was verified in 1925 by R. A. FISHER.[3] Gossett originally tabulated values of $\int_{-\infty}^{z} f(v)dv$ for which $z = .1(.1)3$, and $n = 4(1)10$. Fisher[4] has tabulated values of $t = z\sqrt{(n-1)}$ for which

$$\int_{-z\sqrt{(n-1)}}^{+z\sqrt{(n-1)}} f(v)dv = .1(.1).9,\ .95,\ .98,\ .99,$$

and for $n = 1(1)30$.

Mr. Burrau tabulates values of the integral

$$(2) \qquad \int_{-z\sqrt{(n-1)}}^{z\sqrt{(n-1)}} f(v)dv$$

for which $z\sqrt{(n-1)} = 0(.2)5$, and for $n = 4(2)12$ and $\infty$. The case for $n = \infty$ is, of course, equivalent to finding the values of the Gaussian integral $\frac{1}{\sqrt{2\pi}}\int_{-t}^{t} e^{-\frac{1}{2}z^2}dz$ for $t = 0(.2)5$.

He then observes that since the mean value of $z^2$, i.e. $E(z^2)$, is $1/(n-3)$, thus making $E[(n-1)z^2] = 1$, one might consider the tabulation of the integral

$$(3) \qquad \int_{-z\sqrt{(n-3)}}^{z\sqrt{(n-3)}} f(v)dv$$

for $z\sqrt{(n-3)} = 0(.2)5$, and for $n = 4(2)12$ and $\infty$, rather than the analogous tabulation of (2), with the hope of obtaining values which change relatively little as $n$ changes. The tabulation of (3) is made and the resulting values are considerably more constant with respect to changes in $n$ than the analogous tabulations of (2). For example, the values of (2) and (3) for $z\sqrt{(n-1)} = 2$ and $z\sqrt{(n-3)} = 2$ are:

| $n$ | Value of (2) for $z\sqrt{(n-1)} = 2$ | Value of (3) for $z\sqrt{(n-3)} = 2$ |
|---|---|---|
| 4 | 0.861 | 0.959 |
| 6 | 0.898 | 0.950 |
| 8 | 0.914 | 0.950 |
| 10 | 0.923 | 0.950 |
| 12 | 0.929 | 0.954 |
| $\infty$ | 0.954 | 0.954. |

This indicates that the "significance level" corresponding to $\pm z\sqrt{(n-3)}$ is approximately 95 per cent for all values of $n \geqq 4$. The author does not discuss the problem for $n < 4$.

S. S. W.

[1] W. S. GOSSETT (Student), (a) "The probable error of a mean," *Biometrika*, v. 6, 1908, p. 1–25; (b) *Collected Papers*, Cambridge, 1942, p. 11–34.

[2] The $z$ being used here is that originally used by "Student," and is being used throughout this review. Mr. Burrau, although actually writing about "Student's" problem, unfortunately denoted $z/n^{\frac{1}{2}}$ (in "Student's" sense) by the letter $z$.

[3] R. A. FISHER, "Applications of 'Student's' distribution," *Metron*, v. 5, no. 3, 1925, p. 90–104.

[4] R. A. FISHER, *ibid.*; the table has also been published in Fisher's *Statistical Methods for Research Workers*, eighth ed., Edinburgh, Oliver & Boyd, 1941, p. 167.

**286[K, M].**—A. H. R. GRIMSEY, "On the accumulation of chance effects and the Gaussian frequency distribution," *Phil. Mag.*, s. 7, v. 36, Apr. 1945, p. 295. 16.8×25.3 cm.

There are Tables of $P_n$, $Q_n$, $U_n$, for $n = [1(1)12; 8D]$, exact rational and decimal values being both given for $P_n$, where

$$P_n = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin v}{v}\right)^n dv, \quad Q_n = [6/(\pi n)]^{\frac{1}{2}}, \quad U_n = \frac{2}{\pi}\int_0^\infty \frac{(\sin v)^{n+2}}{v^n}dv.$$

$Q_n$ are the values to which $P_n$ approximates.

Mr. Grimsey's communication was inspired by misprints in L. SILBERSTEIN, "The accumulation of chance effects and the Gaussian frequency distribution," *Phil. Mag.*, s. 7, v. 35, 1944, p. 395–404, and in particular by his values of $P_n$ for $n = 1(1)6$, accompanied by the statement "so far as we know, these integrals have not been previously determined," and by the suggestion that mechanical quadrature would give the values for larger $n$. Mr. Grimsey refers to detailed discussion bearing on this question in J. EDWARDS, *A Treatise on the Integral Calculus*, v. 2, London, 1922, p. 204–212, where the values of $P_n$

are incidentally given for $n = 1(1)7$, that for $P_6 = 11/20$ or $88/160$, occurring in place of $83/160$ given by Silberstein on p. 399.

R. C. A.

**287[L].**—MURLAN S. CORRINGTON, "Tables of Bessel function $J_n(1000)$," *J. Math. Phys.*, v. 24, Nov. 1945, p. 144–147. 17.3×26.3 cm. Compare MTE **74**, v. 2, p. 47–48; the Meissel errors are again noted here.

*Quotations:* A table of Bessel functions for $J_n(1000)$ is given for $n = [935(1)1035; 8D]$, also a graph of $J_n(1000)$ for $930 < n < 1040$. All computations were made to 11D and later rounded off to 8D.

When a frequency-modulated radio wave is analyzed into a spectrum, the carrier and side-frequency amplitudes are proportional to Bessel functions of the first kind, with order equal to the number of the side frequency and argument equal to the modulation index, in accord with the following equation:

$$\sin(w_0 t + (D/\mu)\sin 2\pi\mu t) = \sum_{n=-\infty}^{\infty} J_n(D/\mu)\sin(w_0 + 2\pi n\mu)t$$

where

$w_0$ = carrier frequency, radians per second,
$t$ = time in seconds,
$D$ = deviation in cycles per second,
$\mu$ = audio frequency in cycles per second.

According to the present standards for commercial broadcasting, the deviation can be as much as 75,000 cycles per second and the audio frequency can be as low as 30 cycles per second, so the maximum value of the modulation index, $D/\mu$, is 2500. The table in this article was computed to determine the relative side frequency amplitudes near the edge of the band for a modulation index of 1000. The graph shows that when the side frequency number $n$ exceeds 1000, the amplitudes decrease very rapidly towards zero.

**288[L].**—J. COSSAR & A. ERDÉLYI, *Dictionary of Laplace Transforms.* Admiralty Computing Service, Department of Scientific Research and Experiment, London; Part 3A, no. SRE/ACS. 102, Nov. 1945, 29 leaves. 20.2×33 cm. Mimeographed on one side of each leaf. These publications are available only to certain Government agencies and activities.

This penultimate part of the *Dictionary* is section VII. Section VI (Parts 2A, 1944; 2B, 1945) was a classification according to $f(t)$, the Laplace transform of $f(t)$ being defined by the relation

$$\phi(p) = \int_0^\infty e^{-pt} f(t)\,dt = L\{f(t); p\};$$

see *MTAC*, v. 1, p. 424–425. Part 3A is a Table of inverse Laplace Transforms, that is, a classification according to $\phi(p)$ for the Elementary Functions, Gamma and Related Functions; and Functions defined by Integrals—as in Part 2A. Presumably Part 3B will classify according to $\phi(p)$ the other functions listed on p. 425.

R. C. A.

**289[L].**—RUFUS ISAACS, "Airfoil theory for flow of variable velocity," *J. Aeronautical Sci.*, v. 12, 1945, p. 117. 20.5×28.6 cm.

It is stated that Theodorsen's tabulation[1] of the function

$$C(x) = H_1^{(2)}(x)/[H_1^{(2)}(x) + iH_0^{(2)}(x)]$$

is incorrect for $x \leqslant .1$ and so a new table is given for the real and imaginary parts of $C(x) = F + iG$, for $x = 0(.02).1(.05)4$. There are 3D for $F$, 4D for $G$.

H. B.

[1] THEODORE THEODORSEN, "General theory of aerodynamic instability and the mechanism of flutter," NACA, *Techn. Reports*, no. 496, 1934, p. 418, 426; see *MTAC*, v. 1, p. 404.

IN MEMORIAM NOTE: This review was included in a letter written to me on January 20, 1946, the day on which Harry Bateman left Pasadena, California, by train for New York, where he was to have received notable honors. On the following morning his rare spirit took flight, as a result of a coronary thrombosis. The loss to scholarship, to our Committee, and to *MTAC*, of this extraordinarily able, profoundly learned, and ever helpful intellectual, is irreparable.

H. B. was born in Manchester, England, in 1882. He graduated A.B., as senior wrangler, at Trinity College, Cambridge, in 1903, and was a Smith's Prizeman in 1905. During 1905–1911 he was a fellow of Trinity and received his A.M. in 1906. He was a student at Göttingen and Paris 1905–06, a lecturer at the University of Liverpool 1906–07, and a reader of mathematics and physics at the University of Manchester 1907–10. Coming to the United States in 1910 with 50 publications already to his credit (they included the notable report on Integral Equations), he was for two years a lecturer at Bryn Mawr College. During 1912–15 he was a Johnston scholar at The Johns Hopkins University (Ph.D. 1913), and was lecturer there from 1915 until 1917, when he was appointed professor of mathematics, physics, and aeronautics at the California Inst. Tech. He was elected a fellow of the American Philosophical Society in 1924, of the Royal Society of London in 1928, and of the National Academy of Sciences in 1930, and a vice-president (1935) and Gibbs lecturer (1943) of the American Mathematical Society. His published books were *The Mathematical Analysis of Electrical and Optical Wave-Motion on the Basis of Maxwell's Equations*, Cambridge, 1915; *Differential Equations*, London, 1918; *Partial Differential Equations of Mathematical Physics*, Cambridge, 1932; new edition, New York, 1944. He edited the lectures of H. A. LORENTZ, *Problems of Modern Physics*, Boston, 1927. He was also a collaborator on three published reports of the National Academy of Sciences: (a) Committee on Electrodynamics of Moving Media, 1922; (b) Committee on Hydrodynamics, 1932; (c) Committee on Numerical Integration. His report on tables of Bessel functions was published in *MTAC*, no. 7, 1944, and our Committee had hoped soon to receive his manuscripts of reports on tables of Higher Functions and Integrals, and of a great work on Definite Integrals, replacing the antiquated volume of BIERENS DE HAAN; compare *MTAC*, v. 1, p. 322. About 1900 H. B. participated in a chess tournament between England and America; and prior to 1911 he beat Steinmetz in a tournament. For other details see *Who's Who 1945*, London; *Who's Who in America 1944–45; Amer. Men of Science*, seventh ed., 1944; *Who's Who in Engineering, 1941; Who's Who in Aviation . . . 1942–43; Pan Pacific Who's Who, . . . 1940–41 edition*, Honolulu, Hawaii, 1941; *Leaders in Education*, second ed., 1941; C. M. NEALE, *The Senior Wranglers of the University of Cambridge*, Bury St. Edmunds, 1907; "Poggendorff," v. 5, 1925, and v. 6, 1936.

R. C. A.

**290[L].**—NYMTP, *Table of Characteristic Values of Mathieu's Differential Equation*. Prepared for the Applied Mathematics Panel of the National Defense Research Committee. No. AMP Report 165.1 R. New York, September, 1945. xxiv, 39 p. 20×28 cm. Produced by photo-offset from typescript. These tables are available only to certain Government agencies and activities.

The aim of the volume is to provide an 8D table of the characteristic values of Mathieu's differential equation, for orders up to 15, interpolable in the parameter $s$ (defined below), within the range given.

An account of previous tables of these values has been given by the reviewer in *MTAC*, v. 1, p. 409–419.

Mathieu's differential equation is written in the form

$$d^2y/dt^2 + (b - s\cos^2 t)y = 0. \tag{1}$$

For a prescribed value of $s$ this equation has a solution admitting the period $2\pi$ in $t$ only when $b$ has one of a discrete but infinite sequence of "characteristic values." The periodic solutions fall naturally into two classes, according as they are even or odd in $t$, and the corresponding characteristic values are denoted respectively by $be_r$ and $bo_r$, the suffix denoting the order in the sequence arranged according to increasing numerical magnitude. The ranges included are $r = 0(1)15$, $0 \leqslant s \leqslant 100$; the interval varies, being chosen so that the maximum attainable accuracy is obtained by the use of modified second differences, and these differences are provided. In detail

| be, | | bo, | |
|---|---|---|---|
| $r$ | $s$ | $r$ | $s$ |
| 0 | 0(.2)20(.5)37(1)100 | 1(1)4 | 0(.5)100 |
| 1 | 0(.2)15(.5)50(1)100 | 5(1)7 | 0(1)100 |
| 2 | 0(.2)40(.5)77(1)100 | 8, 9 | 0(2)100 |
| 3 | 0(.2)13(.5)56(1)100 | 10(1)15 | 0(4)100 and 10(20)90 |
| 4, 5, 6 | 0(.5)100 | | |
| 7, 8 | 0(1)100 | | |
| 9, 10 | 0(2)100 | | |
| 11(1)15 | 0(4)100 and 10(20)90 | | |

A table of the Everett interpolation coefficients, $E_2(p)$ and $F_2(p)$, for $p = [0(.001)1;\ 6D]$, is given.

A graph showing the characteristic values as functions of $s$ is reproduced on p. xxiv of the introduction.

It should suffice to say that the NYMTP has carried out the task assigned to it in the efficient manner we have come to expect from this Project. It is clear from the introduction that considerable advances have been made in the technique for the routine computation of these characteristic values, although the new methods do not become really powerful until several decimals of the characteristic value have been determined. In this respect the computer derives, from the small interval, advantages in extrapolation, commensurate with those in interpolation which he confers on the user. Despite some adverse criticism to be given below, the Table is to be welcomed, since any table is infinitely better than no table, and because this seems to be the first table of any magnitude which is easily interpolable in the parameter, and in which provision for interpolation is made. It is also fundamental, since a knowledge of the characteristic values is necessary for the calculation of the Fourier coefficients, and hence of the periodic functions themselves.

On the evidence available to us, however, it does appear that the NYMTP was upon some points ill advised (or instructed). Through ignorance or perversity the judgment of MATHIEU, HEINE, WHITTAKER, GOLDSTEIN, INCE, and nearly every other worker in the field, has been ignored or disregarded by choosing the form (1) of the differential equations, a form which is inconvenient in so many ways both theoretically and practically, and which seems to have only one trivial circumstance in its favor. Indeed, the explanation of the methods of computation and the formulae given in the introduction, and the computations themselves, are all relative to the equation

$$d^2y/dt^2 + (\alpha - 2\theta \cos 2t)y = 0 \tag{2}$$

in which, of course, the parameters are related to those in (1) by

$$\alpha = b - \tfrac{1}{2}s, \quad \theta = \tfrac{1}{4}s. \tag{3}$$

The superiority of (2) over (1) as a canonical form seems to be almost universally recognized.

Fundamentally this rests upon the simplicity and symmetry of the relations between solutions for positive and negative values of $\theta$ (or $s$). For this, reference may be made to *MTAC*, v. 2, p. 1–11. Changing the sign of $\theta$ in (2) leaves $\alpha$ unchanged, and is equivalent to replacing $t$ by $\pm \frac{1}{2}\pi \pm t$. Had $\alpha$ and not $b$ been tabulated, the table would have served equally well for negative and positive values of $s$, but as it is, it is necessary, in order to derive the characteristic number corresponding to a negative value of $s$, to subtract $|s|$ from the value corresponding to $|s|$.

This point can be seen more immediately and strikingly if one compares the elegant symmetry of the graph of characteristic values against $\theta$ (or some equivalent parameter) as given by INCE,[1] STRUTT,[2] LUBKIN & STOKER,[3] MCLACHLAN,[4] and others, with the diagram on p. xxiv of the present volume, imagined carried into the region of negative $s$.

Moreover, we are so far given only characteristic values. If the Fourier coefficients of the periodic solutions are needed—essential before the functions themselves can be calculated—then all the formulae are most conveniently expressed (as in the introduction!) in

terms of $\alpha$. All the indications go to show that NYMTP did, in fact, compute $\alpha$, and then converted to $b$—and that in computing the tables of the corresponding Fourier coefficients (which are promised) it will be $\alpha$ and not $b$ that NYMTP will actually use.

It is fervently to be desired that information equivalent to that given in these tables may before long be made more generally available, but we urge, with all the force at our command, that before this is done the tables should be reconverted so as to give $\alpha$ and not $b$.

In one respect, perhaps, equation (1) gives a lead that was no longer followed. It indicates more clearly than does (2) that the parameter $s$ is generally connected with the square of some significant physical quantity. This is an argument for the retention of the parameter $c$ ($s = c^2$) of some earlier tables.[5]

W. G. BICKLEY

Imperial College of Science and Technology, London

[1] E. L. INCE, "Researches into the characteristic numbers of the Mathieu equation," R. So. Edinburgh, *Proc.*, v. 46, 1925, p. 28.

[2] M. J. O. STRUTT, *Lamésche, Mathieusche, und verwandte Funktionen in Physik und Technik*, Berlin, Springer, 1932, p. 24.

[3] S. LUBKIN & J. J. STOKER, "Stability of columns and strings under periodically varying forces," *Quarterly Appl. Math.*, v. 1, 1943, p. 221.

[4] N. W. MCLACHLAN, "Computation of the solution of Mathieu's equation," *Phil. Mag.*, s. 7, v. 36, 1945, p. 404.

[5] STRATTON, MORSE, CHU & HUTNER, *Elliptic Cylinder and Spheroidal Wave Functions*, New York, Wiley, 1941; compare *MTAC*, v. 1, p. 157f, 414.

**291[L].**—NYMTP, *Tables of Associated Legendre Functions*. New York, Columbia University Press, 1945, xlvi, 303 p. 18.6×26.5 cm. Reproduced by a photo-offset process. $5.00.

Another important volume has been added to the impressive list of publications of the NYMTP. This work, devoted to the associated Legendre functions of first and second kind, is a very useful addition to the already extensive number of tables which have appeared in this area of functional computation as described earlier in *MTAC*, v. 1, p. 116–119. The present table "was begun in 1940, to meet urgent needs for a table to about six significant figures at intervals of 0.1. At this interval in the argument, interpolation in these functions is not everywhere satisfactory; a very considerable amount of further work would be required, in order to make all these functions reasonably interpolable over the entire range covered." The Project has on its agenda the subtabulation of the functions to a more satisfactory interval of the argument.

By the associated Legendre functions of $m$th order and $n$th degree are meant the functions $P_n^m(x)$ and $Q_n^m(x)$, the former called the first kind and the latter the second kind. In the range $x^2 \leqq 1$, it is customary to define these functions as follows:

$$P_n^m(x) = (1 - x^2)^{\frac{1}{2}m} \frac{d^m P_n(x)}{dx^m}, \quad Q_n^m(x) = (1 - x^2)^{\frac{1}{2}m} \frac{d^m Q_n(x)}{dx^m},$$

where $P_n(x)$ is the $n$th Legendrean polynomial, and $Q_n(x)$ the $n$th Legendrean function of second kind, defined as follows:

$$Q_n(x) = \tfrac{1}{2} P_n(x) \log \frac{1 + x}{1 - x} - Z_n(x),$$

$$Z_n(x) = \frac{2n - 1}{1 \cdot n} P_{n-1}(x) + \frac{2n - 5}{3(n - 1)} P_{n-3}(x) + \frac{2n - 9}{5(n - 2)} P_{n-5}(x) + \cdots.$$

It should be observed, however, that H. TALLQVIST, in his tables, defines $Q_n(x)$ as twice the value of the one given.

In the range $x^2 > 1$, it is customary to change the multipliers of the derivatives of $P_n(x)$ and $Q_n(x)$ to $(x^2 - 1)^{\frac{1}{2}m}$ and to replace in the definition of $Q_n(x)$, the function $\log [(1 + x)/(1 - x)]$ by $\log [(x + 1)/(x - 1)]$.

In all previous computations of the Legendrean functions, the range of the independent variable has been restricted to the interval $-1 \leqq x \leqq +1$. The present work thus makes a fundamental contribution, since most of the tables are computed over ranges which lie outside the unit interval.

The present volume contains 19 tables which we may describe as follows:

In the first ten tables the values of $m$ and $n$ are taken over the range $n = 1(1)10$; $m = 0(1)4$, $m \leqq n$, and the computation is carried to 6S. **I.** $P_n^m(\cos\theta)$, $\theta = 0(1°)90°$. **II.** $dP_n^m(\cos\theta)/d\theta$, $\theta = 0(1°)90°$. **III.** $P_n^m(x)$, $x = 1(.1)10$. **IV.** $dP_n^m(x)/dx$, $x = 1(.1)10$. **V.** $(-1)^m Q_n^m(x)$, $x = 1(.1)10$. **VI.** $(-1)^{m+1} dQ_n^m(x)/dx$, $x = 1(.1)10$. **VII.** $i^{-n}P_n^m(ix)$, $x = 0(.1)10$. **VIII.** $i^{-n}dP_n^m(ix)/dx$, $x = 0(.1)10$. **IX.** $i^{n+2m+1}Q_n^m(ix)$, $x = 0(.1)10$. **X.** $i^{n+2m-1}dQ_n^m(ix)/dx$, $x = 0(.1)10$.

Of the remaining nine tables in the volume, the first four give the values of the functions and their first derivatives at fractional half orders, that is to say, for the subscript $n + \frac{1}{2}$, $n = -1(1) + 4$. In these tables $m = 0(1)4$. The tables may then be described as follows:

**XI.** $P_{n+\frac{1}{2}}^m(x)$, $x = [1(.1)10$; 4S–6S]. **XII.** $dP_{n+\frac{1}{2}}^m(x)/dx$, $x = [1(.1)10$; mainly to 6S]. **XIII.** $(-1)^m Q_{n+\frac{1}{2}}^m(x)$, $x = [1(.1)10$; 6S]. **XIV.** $(-1)^{m+1} dQ_{n+\frac{1}{2}}^m(x)/dx$, $x = [1(.1)10$; 6S].

The remaining five tables are supplementary and are contained in 10 pages. They may be described as follows:

**XV.** Exact values of $P_n^m(x)$ and $dP_n^m(x)/dx$, $m = 0, 2, 4$; $n = 4$ or $5(1)10$, $x = 1(.1)2$.

**XVI.** Exact values of $P_n^m(x)/(x^2-1)^{\frac{1}{2}m}$ and of $\left[\frac{d}{dx}P_n^m(x)\right]\Big/(x^2-1)^{\frac{1}{2}m-1}$, $m = 1, 3$, $n = 2$ or $3(1)10$, $x = 1(.1)2$. **XVII.** Values of $(x^2-1)^{\frac{1}{4}}P_{n+\frac{1}{2}}^m(x)$ and of $(x^2-1)^{\frac{1}{4}}dP_{n+\frac{1}{2}}^m(x)/dx$, $m = 1, 3$, $n = -1(1)4$, $x = [1(.1)2$; 4D–7D]. **XVIII.** Values of Legendre's normalizing factor, $N_n^m = \{2(n+m)!/[(2n+1)(n-m)!]\}^{\frac{1}{2}}$, $n = 0(1)10$, $m \leqq n$, 10S. **XIX.** First eleven coefficients in the expansion of $CF(m+\frac{1}{2}; \frac{1}{2}-m; n+\frac{3}{2}; -t)$, where $C$ is equal to $[\Gamma(\frac{1}{2})\Gamma(n+m+1)/[\sqrt{2}\Gamma(n+\frac{3}{2})]]$.

Except in the case of $P_n^m(\cos\theta)$ and its derivative, which were computed directly from their expansions in terms of $\sin p\theta$ and $\cos p\theta$, the other tables were evaluated by means of standard recurrence formulae. In the final checking of the computation the following formula was used:

$$P_n^m(x)\frac{d}{dx}Q_n^m(x) - Q_n^m(x)\frac{d}{dx}P_n^m(x) = \frac{-(-4)^m\Gamma[\frac{1}{2}(n+m+1)]\Gamma[\frac{1}{2}(n+m+2)]}{(x^2-1)\Gamma[\frac{1}{2}(n-m+1)]\Gamma[\frac{1}{2}(n-m+2)]}.$$

In the case of $P_n^m(\cos\theta)$, when the values were checked against those published by Tallqvist, no errors were discovered.

In addition to the tables, a number of formulae are developed which were found useful in the computation of preliminary values. A bibliography of 47 references is also given, which provides an adequate survey of the present status of computation in this active field.

H. T. D.

**292[L, M].**—PIERO GIORGIO BORDONI, "The conical sound source," Acoustical Soc. Amer., *J.*, v. 17, Oct. 1945, p. 125.

$$\int_0^{2\pi} e^{iz\cos\phi}d\phi = \int_0^{2\pi} e^{-iz\cos\phi}d\phi = 2\pi T_0(z),\ G_n(z) = z^{-(n+2)}\int_0^z t^{n+1}T_0(t)dt.$$

There is a table of $G_n(z)$, for $n = 0(1)6$, $z = [0(1)5$; 3D–4D].

**293[L, M].**—GREAT BRITAIN, NAUTICAL ALMANAC OFFICE, *Tables of the Integrals* $C(t) = k^{\frac{1}{2}}\int_0^\infty e^{-u}\cos(2tk^{\frac{1}{2}}u - ku^2)du$, $S(t) = k^{\frac{1}{2}}\int_0^\infty e^{-u}\sin(2tk^{\frac{1}{2}}u - ku^2)du$, Department of Scientific Research and Experiment, Admiralty Computing Service, September 1945. No. SRE/ACS. 90. 11 p. 20.1 × 30.8 cm. This publication is available only to certain Government agencies and activities.

Tables to 3D are given for $C$, $S$, and $(C^2 + S^2)^{\frac{1}{2}}$, for the following values of $k$ and $t$:

$k = \frac{1}{4}$, $t = -10, -3(.1) + 3, 10$;
$k = \frac{1}{2}$, $t = -8(1) - 5(.5) - 3(.1) + 8$;
$k = 1$, $t = -6, -5(.5) - 3(.1) + 6$;
$k = 2, 4$, $t = -5(.5) - 3(.1) + 7$;
$k = 8$, $t = -5(.5) - 3(.1) + 9$.

"Graphical accuracy only was required and the method of computation was such that the last of the three decimals tabulated may be in error by a few units." There is a graph of $(S^2 + C^2)^{\frac{1}{2}}$ for $-5 < t < +6$.

The differential equations satisfied by $S(t)$ and $C(t)$ are

$$dS/dt = 2\pi C - S/k^{\frac{1}{2}}, \quad dC/dt = 1 - 2\pi S - C/k^{\frac{1}{2}},$$

for the range of values $-3 < t < +3$; $S(t)$ and $C(t)$ were obtained by numerical integration of these equations. For larger values of $t$, $S(t)$ and $C(t)$ were computed from certain appropriate asymptotic trigonometric series.

R. C. A.

**294[M].**—W. HEISENBERG, "Die beobachtbaren Grössen in der Theorie der Elementarteilchen. II.," *Z. f. Physik*, v. 120, 1943, p. 702.

There is a short table of

$$p_n = \frac{1}{\pi}\int_0^\infty dx(2x)^n \frac{d}{dx} \arctan\left(\frac{4}{\sqrt{\pi}}\int_0^{x^{\frac{1}{2}}} ds\, s^{\frac{1}{2}}e^{s^2}\right)$$

$p_1 = 1$, $p_2 = 3$, $p_3 = 13$, $p_4 = 79$, $p_5 = 641$, $p_6 = 6579$, etc. It is, of course, remarkable that these integrals are integers.

H. B.

**295[U].**—HAMBURG, DEUTSCHE SEEWARTE, publication no. 2154, *F-Tafel. Tafel zur vereinfachten Berechnung von Höhenstandlinien*. 3 Auflage. Hamburg, August, 1941. xxiii, 88 p. 19.6 × 29.2 cm. In the third edition there were extensions and corrections of the introductory material, and of 8 of the 11 tables.

The method and principal table of this volume are similar in many respects to those of H. O. 208 (DREISONSTOK, see *MTAC*, v. 1, p. 79f). The astronomical triangle is divided into two right spherical triangles by a perpendicular from the zenith upon the hour circle of the star; $U$ is the co-declination of the foot of the perpendicular, and $V$ is log cos $B$, where $B$ is the angle subtended at the zenith by $U$. By Napier's rules,

$$\tan U = \cos t \cot L$$

and

$$\sin B = \sin t \cos L,$$

where $t$, $L$, and $d$ are the local hour angle, latitude and declination respectively. By applying another of Napier's rules to the right triangle of which the star is one vertex, the altitude, $h$, may be found by

$$\sin h = \cos B \sin (d + U)$$

or

$$\log \sin h = V + \log \sin (d + U).$$

For the determination of azimuth, $Z$, two more auxiliary quantities are introduced, $P$ which is the great circle distance from the star to the east- or west-point of the horizon, and Gr. $\delta$ which is the declination of the intersection of the hour circle of the star with the prime vertical. Thus, $\sin t \cos d = \cos P$ and $\sin Z = \cos P \sec h$. Also, tan Gr. $\delta = \tan L \cos t$.

In **Table F I**, with vertical argument, latitude 0(1°)70°, and horizontal argument, local hour angle $0(4^m)6^h$, three values per page, there are tabulated four quantities, $U$ to the nearest 0'.1, $V$ to 5D, Gr. $\delta$ and $P$, each to the nearest 0'.1. In the second part of Table F I, the vertical argument is latitude, 70°(1°)90°, and the horizontal argument is local hour angle $0(4^m)6^h$, nine values per page, three in each horizontal section.

At the bottom of the vertical columns in Table F I are azimuths; entering the left hand column with altitude as argument, and moving across the pages horizontally until one finds under $P$, the value already copied out, one can drop to the bottom of the column and read off the azimuth angle. Since the tabulated values of the azimuth angle go up to 90° only, it is necessary to have another device to determine the quadrant. When the hour angle is greater than $6^h$, the azimuth is measured from the elevated pole; when the local hour angle is less than $6^h$ and $L$ and $d$ are of opposite name, the azimuth is measured from the depressed pole. If the local hour angle is less than $6^h$ and $L$ and $d$ are of the same name, the azimuth is measured from the elevated or depressed pole according as the declination is greater or less than the quantity Gr. $\delta$.

In case the altitude is great, or the azimuth near 90°, the value of the azimuth may be poorly determined by the use of Table F I. In such a case, it will be noted that the value of $P$ lies below a dotted line running across the page. One must then use instead **Table F XI**, which gives $P$ to the nearest minute of arc and the variation in $P$ corresponding to 1' change in $d$ or $h$.

**Table F II** is a table of log sin $x$, $x = [0(0'.1)6°(1')90°$; 5D], with generous tables of proportional parts.

**Tables F III** and **F IV** represent the principal advantages this volume possesses over other similar tables; they permit one to determine the corrections (to the nearest 0'.1) to the computed altitude corresponding to slight changes in time (up to $2^m$ by $10^s$ steps) or latitude (up to 30' by 1' steps) respectively. In both cases, one can interpolate very easily by a shift of the decimal point. **Table F III** is a well-designed triple-entry table occupying only five pages; one starts down the column at the left headed by the value nearest the assumed latitude, stops at the value nearest the computed azimuth and moves to the right to the column headed by the number of seconds change in time. **Table F IV** is a small double-entry table on a single page; the vertical argument is azimuth 0(5°)20°(2°)90°, and the horizontal argument is change in latitude, 1'(1')10'(10')30'. These two tables allow one to work either with an assumed position or with a dead reckoning position.

**Table F V** is for changing time into angular measure and conversely. **Table F VI** gives the corrections for refraction, semi-diameter and parallax to be applied to the altitude (3°–90°) of the lower- or upper-limb of the moon; there is a supplementary table for height of eye. **Table F VII** gives the combined correction for refraction and height of eye (0–30 meters) to be applied to the altitudes (3°–90°) of fixed stars or planets. **Table F VIII** yields the correction for refraction, semi-diameter and height of eye (0–30 meters) to be applied to altitudes (3°–90°) of the sun's lower limb; there are also two auxiliary tables to provide corrections to the altitudes to take care of the varying semi-diameter of the sun through the year, and for the case where the sun's upper limb was observed. The latter takes only a very small amount of space and would seem to be quite worthwhile. **Tables F IX** and **F X** provide similar corrections for use with the bubble sextant.

The tables are well printed on a good grade of paper. In a number of cases, the rules needed to make decisions as to quadrants, etc. are printed on each page. As for the accuracy of the tabulated values, only a few rounding off errors of a unit in the last place were discovered in a brief examination.

CHARLES H. SMILEY

Brown University

**296[U].**—HYDROGRAPHIC OFFICE, *Publication no.* 351, *v.* 7, *Secret. Celestial Air Navigation Table. Volume* 7, *Latitudes* 60°–70°, Tokyo, November 1942. iii, 320 p. 18 × 25 cm.

This is the seventh volume of a series originally intended to include only six and to cover latitudes 60°S to 60°N. Since its general design and content are similar to those of the first six volumes which have already been reviewed (RMT **273**), it will suffice to point out the only change which has been made. The auxiliary table for changing time measure into angular measure has been replaced by another which has been pasted over the old one. Only the degrees were given in the main body of the old table and one had to move to the right hand edge of the page to pick up the minutes associated with the degrees; in the new table, the degrees and the minutes are given together in each entry.

It was earlier stated, *MTAC*, v. 2, p. 44, that v. 4 was the first of the series to be issued. Actually v. 3 appeared in August, 1940; v. 4 in May, 1941; v. 1 in September, 1941; v. 5 in December, 1941; and v. 6 in February, 1942. A copy of v. 2 has not yet reached the reviewer's hands.

CHARLES H. SMILEY

**297[U].**—HYDROGRAPHIC OFFICE, *Publication No.* 9582. *Secret. Altitude-Azimuth Tables for September* 1945 (*Sun, Moon, Planets and Fixed Stars*) *computed for Kisarazu, Naha, Iwo Jima and Kanoya and for Japanese Central Standard Time.* Tokyo, Japanese Hydrographic Office, August 1945, 42, vi, viii p., 18.2 × 25.7 cm.

These tables were designed for celestial air navigation over a limited region in the western Pacific. All descriptive material, explanations, star names, etc., are in Japanese characters; tabular material is in Arabic numerals. One volume is prepared for each month and contains all the material needed for that month. For each 20 minutes throughout the 24 hours of Japanese Central Standard Time (time zone −9), altitudes and azimuths of the celestial bodies, most useful to navigators, as computed for four locations, namely, Kisarazu near Tokyo, Naha on Okinawa, Iwo Jima, and Kanoya near Kagoshima, are given to the nearest minute of arc and to the nearest degree respectively. For the sun, the altitude ($> 4°$) is given with refraction for 4000 meters elevation added. In the case of the moon, the altitude (also $> 4°$) with corrections added for refraction at 4000 meters and parallax, is given. For bright stars and planets, the altitudes (occasionally down to 2° and always with refraction for 4000 meters), are also given.

Ten pages are used in presenting material for each of the four geographical positions listed above. Each page carries three vertical sections, each of which contains three double columns and includes data for a single day. The double column to the left gives the altitude and azimuth (say, 2956 050 which means 29° 56′ altitude, 50° azimuth) for a bright star until sunrise, then for the sun until sunset, and finally for another bright star until midnight. The bright stars in this column are chosen so as to be useful in determining latitude.

The second double column in the vertical section gives the altitude and azimuth of the moon when it is available for observation and for bright stars or planets for the remaining time. The third double column presents the altitude and azimuth of bright stars and planets.

Since only the sun and moon are easily observed in the daytime, one or two double columns would ordinarily be blank, opposite the daylight hours as listed on the left side of the page. These spaces have been filled with data for bright stars and planets, well located for observation in the *night hours;* this material is printed in heavy black type and the time argument is printed on the right side of the page in heavy black type also. Thus there are available for observation, one or two celestial objects during the day and five or four during the night. The planets included in this volume are Venus and Saturn; the bright stars used are Capella, Vega, Alpheratz, Arcturus, Regulus, Altair, Procyon, Rigel, Spica, Sirius, Antares, Fomalhaut and Canopus.

These tables differ considerably from common navigation tables in that much greater differences between the assumed and real positions of the observer are used here, and hence larger differences between computed and observed altitudes are encountered. In one of the examples worked out in the explanation in the back of the book, altitude intercepts of +2° 07′ and −37′ are used in plotting two lines of position and determining a fix. The

inaccuracies introduced in this way would be too large for surface navigation, but may be tolerated in air navigation.

In addition to the principal tables just described, there is one page on which the altitude of Polaris is given for each of the four geographical positions named and for each 20 minutes of the night for every fifth day of the month. There follows a one-page interpolation table, giving the corrections to be applied to tabulated altitudes. The vertical argument is tabular difference in altitude 0(4′)5°, and the horizontal argument is: time interval $0^m.5(0^m.5)10^m$. There follow four pages of tables, one for each of the geographical positions giving corrections to the tabulated altitudes corresponding to other positions near the chosen ones. There are 10 such points near Tokyo, 8 near Iwo Jima, 9 near Kagoshima, and 6 near Okinawa. These positions vary as much as approximately 6° in latitude and 4° in longitude from the central position. It will be realized that these corrections will be most useful in the case of a body observed at low altitude. At the top of a column is the name of a geographical location for which the altitude correction applies. The left-hand column of the page contains values of the argument $Z$ (azimuth of the celestial body) 0(3°)180°; the right-hand column contains values of the argument $Z$ 180°(3°)360°. No provision is made for determining the change in azimuth in going from the standard position to the nearby one.

The remainder of the volume is devoted to explanations; an expert navigator may follow these, even though he does not read Japanese, since the numerals are Arabic. One notes that the page of tabular values reproduced for use in working the examples covers April 22–24, 1945, and that several improvements have been made in the arrangement of the tables since that time, particularly in the manner in which heavy black type is used to set apart night-time values printed opposite day-time hour arguments.

Since much of the air travel over oceans will in the future be over a limited number of well-defined paths, it may be that the type of table described here may come into use over such paths. The advantage of dispensing with the need of an almanac and the rapidity with which a fix can be determined may be considered as outweighing in *air* navigation, the inaccuracies introduced by the necessarily greater differences between dead reckoning position and assumed position. There remains the question as to whether or not it will be economically profitable to prepare such tables, no matter how convenient and rapid they may be, if their useful life is only one month.

CHARLES H. SMILEY

**298[U].**—P. V. H. WEEMS, *Line of Position Book*, fourth ed. Annapolis, Md. Weems System of Navigation, 1943, viii, 56 p. 14.5 × 24.3 cm. $2.00.

This book, in its first edition issued in 1927 by the U. S. Naval Institute, contained tables useful for latitudes 65° South to 65° North and was intended for surface navigation. The second edition appeared in 1928 and extended its field to include all latitudes; it also contained simplified procedures for aerial navigation. The third edition followed after an interval of 12 years. The author is now a Captain (retired) of the U. S. Navy and has long been well known for his contributions to the art and science of navigation. The tables in this volume are said to provide altitudes accurate to the nearest minute of arc without interpolation; greater accuracy can be had by interpolating.

There are three principal tables, Table A, Table B, and Table A-Polar. Table A has as vertical argument, latitude 0(1°)65°, and as horizontal argument, local hour angle, 0(1°)180° by integral degrees. The values, 0 to 90° of local hour angle are found at the tops of pages, 5° per page except the first page which has 6°, and the tabulated quantities are $A$ and $K$; local hour angles 90° to 180° are found at the bottoms of pages, and $A$ and $(180° - K)$ are tabulated. In the notation of RMT **103**, $A$ is $10^5 \log \sec a$ and is given to one decimal place for local hour angles 0 to 20°, and 160° to 180° to the nearest integer for local hour angles 20° to 160°. $K$ is the declination of the foot of the perpendicular from the zenith upon the hour circle through the celestial body and is given to the nearest tenth of a minute of arc throughout.

Table B is a table of $10^5 \log \sec x$ for $x = 0(1')90°$ when the argument at the top and left of the page is used, or a table of $10^5 \log \csc x$ for the same interval when the argument

at the right and bottom of the page is used. Values less than 665 are given to one decimal, other values to the nearest integer.

Tables A and B are credited to S. OGURA, hydrographic engineer of the Japanese Navy.

Table A-Polar is similar to Table A, except that latitudes 60° to 90° are covered and values of A less than 665 are given to one decimal, other values of A to the nearest integer.

Weems suggests that the formula $B(h) = A + B(K \sim d)$ be used for the determination of the altitude and that the azimuth diagram of ARMISTEAD RUST, USN Captain (retired), be used to determine the azimuth. This diagram is reproduced on a scale 1.5 times the original on two pages, one of which is folded. A diagram enabling one to determine the hour angle of a celestial body on the prime vertical takes care of the ambiguous case in the determination of the quadrant of the azimuth angle arising when the latitude is of the same name and greater than the declination.

Auxiliary tables given in the volume are those providing corrections for observed altitudes obtained with ordinary sextants and with bubble sextants. A brief but adequate explanation of the use of the tables is given and several examples are worked out.

In the preface to the third edition, the author says, "The Line of Position Book has stood the test of time. No errors have been noted in the Altitude Tables. Except for the method of H.O. 214, it remains the shortest, easiest method for working the line of position." In this statement, the author fails to mention one of the strong points of his method, namely that it works in all latitudes and for all altitudes. All of the modern tables with which a sight can be reduced in appreciably less time than with the *Line of Position Book* are limited methods, usually limited in both latitude and altitude.

Perhaps the outstanding criticism to be made of this book of tables is that local hour angle changes from page to page rather than latitude. There is a definite advantage in finding all of the data to be used at one time on a single page rather than scattered through the book. By the same argument, the polar table should be amalgamated with Table A. It may be noted here that these changes have been made by the author in his *New Line of Position Tables*, Annapolis, Md., 1944.

CHARLES H. SMILEY

## MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT **286** (Silberstein); N **53** (Furnas, Wrinch); **54** (McIntyre); Q **17** (?).

**79.** *Barlow's Tables of Squares, Cubes, Square Roots, Cube Roots, and Reciprocals, of all Integer Numbers up to* 12500, edited by L. J. COMRIE. Fourth ed., 1941. See *MTAC*, v. 1, p. 16–17.

As a part of the checks applied in the transcription of these tables to punch-cards, the tabular values of $f(n) = \sqrt{n}$, $\sqrt{10n}$, $\sqrt[3]{n}$, and $1/n$ for $n = 1000(1)12500$ were inverted; i.e., $f^{-1}[f(n)]$ was found. Differential corrections accurate to a few additional decimal places were computed. Doubtful cases were further examined. All values were found to be correct, except two which need a correction of a unit in the last place:

1) For $n = 3195$, $\sqrt{10n}$ is tabulated as 178.745630. The correct value is 178.7456293, which rounds off to 178.745629. Also the corresponding differences must be changed to 27975, 27971.

2) For $n = 4575$, $\sqrt[3]{n}$ is tabulated as 16.600851. The correct value is 16.6008516, which rounds off to 16.600852. Also the corrected differences, 1210, 1209.

L. J. CUNNINGHAM

Aberdeen Proving Ground

EDITORIAL NOTE: These errors are also to be found in the third ed. 1930, and in the 1935 pirated ed.

**80.** JAMES BURGESS, "On the definite integral $(2/\pi^{\frac{1}{2}})\int_0^t e^{-t^2}dt \cdots$," See *MTAC*, v. 1, p. 449.

In trying to use Burgess' Table of $L$ as a check on certain computations we found several errors in this table. However, to decimals varying from 11 to 15, we checked Wrench's corrections to Burgess' table except for $t = 6$. There we agreed with Burgess rather than Wrench. In order to settle this point once and for all we made three separate computations, one using the 15th convergent of a continued fraction for $L/2t$, the second using the 16th convergent, and the third using the asymptotic series for $L$. All three were sufficiently accurate to verify Burgess' 15 decimal value and refute Wrench's value. In addition we made use of a trick for increasing the accuracy obtainable from an asymptotic series to compute the value

.98665 31092 31165 44447 for $L$ at $t = 6$.

J. Barkley Rosser
Edward M. Cook
John A. Brittain

Alleghany Ballistic Laboratory
Cumberland, Md. 15. XI. 45

I carefully recalculated $L(6)$, this time using all 24 decimal places of $F(x)$ and its reduced derivatives for $x = 8.4$ and 8.5 as tabulated by Sheppard. The final result is $L(6) = .98665\ 31092\ 31165\ 44446\ 847$ where the error in the 23rd place is less than one unit. This confirms the above result to 20D. In view of our agreement in all other cases this should conclude the checking of the $L$-table.

J. W. Wrench, Jr.

4211 Second St., N.W.
Washington 11, D. C. 20.XI.45

**81.** Olinthus Gilbert Gregory (1774–1841), *Mathematics for Practical Men . . .* second American ed. from the second London edition, Philadelphia, Baird, 1863. In the supplementary tables is, "A Table of Circles, from which knowing the diameters, the areas, circumferences, and sides of equal squares are found." At the conclusion of the tables Gregory states that they "were computed with great care by the author's esteemed friend, the late H. Goodwin, Esq. of Blackheath, a gentleman whose indefatigable perseverance and remarkable accuracy in reference to numerical computations cannot be too highly characterized. They are inserted here to supersede the necessity of consulting some erroneous tables of the areas, etc., of circles recently put into circulation."

These three tables are each for Diameter (D) = [1(.25)100.75; 8D], except that the Area for D = 1 is to 7D. With these same values of D I recalculated the whole of the table for circumferences (C) and found that of Goodwin's 400 tabular values of C, 126 are correct, one is too small by 3 units in the last figure, 22 are deficient by 2 final units, 244 are too small by one unit in the last place, and one is too great by 3 final units. The following major errors were discovered in this table:

| *D* | *For* | *Read* |
|---|---|---|
| 37.75 | 118.59572267 | 118.59512267 |
| 43.00 | 135.08348410 | 135.08848410 |
| 48.00 | 150.79644797 | 150.79644737 |
| 50.25 | 157.96503084 | 157.86503084 |
| 81.75 | 256.82579942 | 256.82519943 |
| 84.00 | 263.89378269 | 263.89378290 |
| 96.25 | 302.37829269 | 302.37829291 |

Stanley M. Shartle

353 South Indiana St.,
Danville, Indiana

EDITORIAL NOTE: The posthumously published "Goodwin" table, discussed by Mr. Shartle, appeared as Table II in the first London edition of Gregory's work, London, 1825, p. 399–406; in the third London edition, 1848, edited, rev., and enl. by Henry Law, it is Table VII in the Appendix, p. 66–72. In each of these editions the seven sets of figures, quoted above as erroneous, are in agreement. Furthermore, the quotation from Gregory concerning the tables is identical with what appears in the first edition. In at least four of the cases cited by Mr. Shartle, errors made by Gregory, in seeing one of the tables through the press, are definitely suggested. "H. Goodwin" is undoubtedly the HENRY GOODWYN to whom we have already referred, *MTAC*, v. 1, p. 22–23, 67, 100, 372. According to the *Gentleman's Magazine*, v. 94, p. 381, he died at Blackheath Sept. 27, 1824, aged 79. The most extensive tables (in decimal places) listed in the FLETCHER, MILLER & ROSENHEAD, *Index*, are as follows: Circumference, for D = [2(2)200; 6D], in L. POTIN, *Formules et Tables Numériques*, Paris, 1925, p. 204–205; Area, for D = [0(.1)100; 6D], in °C. TODD, *A Series of Tables*. Second ed., London, 1853. In Wölffing's *Mathematischer Bücherschatz* we find °M. F. KUNZE, *7-stellige Kreisflächen von 0.01 bis 99.99*. Dresden, 1868; and in Kayser's *Bücher-Lexicon* is the following: °J. MODER, *Tabellen nebst Anweisung zur bequemen Aufsuchung d. Kreisumfangs, d. Kreisfläche, der Oberfläche der Kugel, d. Kubikinhalts der Kugel bis auf 20 Dezimalstellen, sowie zur Aufsuchg. d. Diameters resp. Radius wenn e. der obigen Dimensionen gegeben ist, f. alle Bauhandwerker u. Solche, welche m. Vermessungsarbeiten zu thun haben, sowie zum Selbstunterricht . . .* Wiesbaden, 1886, 30 p.

A copy of the first London edition of Gregory's volume is in the Athenaeum Library, Providence, R. I. The first American edition of Gregory (from the second London edition of 1833) was published at Philadelphia in 1834.

In DeMorgan's article on "Table" in the *English Cyclopaedia, Arts and Science Section*, v. 7, 1861, we find the following in col. 982: "Mr. Goodwyn (of Blackheath) was an indefatigable calculator, and the preceding tables are the only ones of the kind which are published. His manuscripts, an enormous mass of similar calculations, came into the possession of Dr. Olinthus Gregory, and were purchased by the Royal Society at the sale of his [Gregory's] books in 1842." It would be of interest to learn about the contents of Goodwyn's manuscripts, which are now available at Burlington House.

**82.** NYMTP, *Table of Characteristic Values of Mathieu's Differential Equation*, 1945. See *MTAC*, v. 2, RMT **290**.

*For* $r \geqslant 0$, *read* $r > 0$, in the following: p. xiii, l. 10 and 14; p. xiv, l. 5, 10, and 14; p. xv, l. 12.

NYMTP

**83.** N. NIELSEN, *Recherches Numériques sur certaines Formes Quadratiques*. Copenhagen, 1931. xi, 160 p. Tables I–II, p. 1–17.

Errors in T. I–II are indicated in my "A List of errors in a table of numbers $D$ for which $x^2 - Dy^2 = -1$ has solutions in integers," *Nieuw Archief v. Wiskunde*, s. 2, v. 21, 1943, p. 194–196. Table I gives all numbers $D$ under 10000 for which the equation

$$x^2 - Dy^2 = -1 \tag{1}$$

is solvable; Table II gives those $D$'s which are sums of two squares and less than 10000 for which (1) is insolvable.

The following is a complete errata list.

Table I. *Insert:* 2153, 4330, 4777, 5545, 5597, 5669, 6418, 7274, 7330, 8125, 9434, 9442, 9802.
*Delete:* 2157, 2353, 4705, 4786, 5218, 5986, 5989, 6338, 6505, 6605, 7294, 8585, 8725, 9477, 9962.

Table II. *Insert:* 2353, 4546, 4705, 5218, 5986, 5989, 6338, 6505, 6605, 8585, 8725, 9701, 9962.
*Delete:* 4330, 4777, 5545, 5597, 6418, 7330, 8125, 9442, 9802.

This list was obtained by a recalculation and comparison with two other tables:

P. SEELING, "Ueber die Auflösung der Gleichung $x^2 - Ay^2 = \pm 1$ in ganzen Zahlen, wo $A$ positiv und kein vollständig Quadrat sein muss," *Archiv Math. Phys.*, v. 52, 1871, p. 48–49.

° W. PATZ, *Tafel der regelmässig Kettenbrüche für Quadratwurzeln aus den natürlichen Zahlen* 1–10000. Leipzig, 1941. Errata were found in these tables as follows:

Seeling: *Insert* 3802; *delete* 4717.

Patz: The diamond sign is printed one line too low at 6938, 6949, 6953, 9698.

N. G. W. H. BEEGER

Nicolaas Wirsenkade 10,
Amsterdam

## UNPUBLISHED MATHEMATICAL TABLES

References have been made to unpublished tables in RMT **279** (Beeger); MTE **81** (Goodwyn); N **52** (NYMTP); QR **23** (Hillman & Salzer).

**42[F].**—N. G. W. H. BEEGER, *Second List of Additions* to *Binomial Factorisations*, Manuscript table of 5 p. in possession of D. H. L.

The title refers to A. J. C. Cunningham's extensive *Binomial Factorisations* in 9 v., London, 1923–1929. The first list of additions was prepared by Beeger in 1933 and published under the title *Supplement to Binomial Factorisations by the late Lt.-Col. Allan J. C. Cunningham, R.E. vol. I to IX*, by the London Mathematical Society, London, 1933. The present second list gives the factorizations of 81 numbers left unfactored in v. 1 and v. 2 and the 1933 Supplement.

D. H. L.

**43[F].**—HANSRAJ GUPTA, Three ms. tables in possession of the author at Government College, Hoshiarpur, India.

(a) Tables of partitions giving the values of $p(n, m)$, the number of partitions of $n$ into exactly $m$ non-zero summands for values of $n \leqslant 200$. This table gives also the number of partitions of $n$ in which the largest part is exactly $m$. [Compare *MTAC*, v. 1, p. 313–314.—Ed.]

(b) Tables of distributions giving the values of $u(m, a)$, the number of ways in which $m$ different particles can be accommodated in exactly $a$ similar cells. $m \leqslant 50$.

(c) The number of lattice points on the sphere $x^2 + y^2 + z^2 = n$, for values of $n \leqslant 10000$.

H. GUPTA

**44[F].**—H. K. HAMMER, *Tables of Periods of Reciprocals of Primes in Various Number Systems*. Typed mss. prepared by, and in possession of the author at 21 West Street, New York City 6. There are copies of these mss. in the Library of Brown University. 11 leaves + 35 leaves. 21 × 29.5 cm.

Each of the first 11 leaves is devoted to the periods of $1/p$, $p < 100$, in each one of the number systems with base 2(1)12. The number of integers in each period, $t$, varies from 1 to 96. The next 35 leaves contain the periods $1/p$, $p < 1000$ in the decimal notation; $t$ varies from 1 to 982.

H. K. HAMMER

EDITORIAL NOTE: All of the periods listed on the last 35 leaves were included in those published by HENRY GOODWYN in his *A Table of the Circles arising from the Division of a Unit or any other whole number, by all the integers from* 1 *to* 1024, *being all the Pure Decimal Quotients that can arise from this source*. London, 1823. Tables of the period of $1/p$, for $p < 1000$, were given by K. F. GAUSS, in his *Werke*, v. 2, 1863, second ed., 1876, p. 412–434. A table of periods of $1/p$ to the base 2, for $p \leqslant 383$, has been given by G. BELLAVITIS, in R. Accad. Naz. d. Lincei, *Cl. d. Sci. fis., mat., e nat., Memorie*, s. 3, v. 1, 1877, p. 790–794.

A. J. C. CUNNINGHAM, "On binal fractions," *Math. Gazette*, v. 4, 1908, p. 266–267, gives the complete period of $1/n$ to base 2, for $n < 100$; and tables to the same extent for the bases 3 and 5, in "On tertial, quintal, etc. fractions," *Math. Gazette*, v. 6, 1911, p. 111–116. The first 16 digits in the period of $1/n$, for $n = 1(1)1728$, to base 12, may be obtained from G. S. TERRY, *Duodecimal Arithmetic*, London, New York, Toronto, Longmans, 1938, p. 59–70.

**45[G, I].**—H. E. SALZER, *Coefficients in Polynomials.* Two mss. in the possession of the author at NYMTP.

The first ms. gives the coefficients of the various powers of $p$ in the polynomials $q(q^2 - 1^2)(q^2 - 2^2) \cdots (q^2 - n^2)$ where $q = 1 - p$, and $n = 1(1)10$.

The second ms. gives the exact values of the coefficients of the eleventh to the twentieth Laguerre polynomials.

H. E. SALZER

EDITORIAL NOTE: For Laguerre polynomials compare *MTAC*, v. 2, p. 31.

## MECHANICAL AIDS TO COMPUTATION

**18[Z].**—MAX BORN, R. FÜRTH & R. W. PRINGLE, "A photoelectric transformer," *Nature*, v. 156, 22 Dec. 1945, p. 756–757.

The following is the first paragraph of this preliminary account of the transformer, from the department of mathematical physics, University of Edinburgh: "The 'Fourier transform' $g(y)$ of a function $f(x)$, usually defined by the integral

$$g(y) = \frac{1}{\sqrt{2\pi}} \int_a^b f(x) e^{-iyx} dx,$$

plays an important part in many problems of pure and applied physics. It represents, for example, the connexion between the intensity distribution of a wave scattered by matter of a certain density distribution, which has to be calculated in a number of acoustical and optical problems and, above all, in X-ray crystal analysis work. It further allows the resolution of a complicated oscillation into a continuous frequency spectrum of harmonic oscillations, which is required in many problems of mechanical and electrical engineering. It therefore seems of some importance to have an instrument by which the Fourier transform of a given function can be automatically and quickly produced. We have now succeeded in building up an instrument which produced the graph of the function

$$g'(y) = \int_a^b f(x) \cos(yx + \delta) dx$$

on the screen of a cathode ray oscillograph, from a mask cut out of black paper in the shape of the graph of the function $f(x)$, or from a record of this function on a plate or film in density variation. Obviously two of the functions $g'(y)$ for two values of $\delta$, say, $\delta = 0$ and $\delta = \frac{1}{2}\pi$, are equivalent to the complex function $g(y)$ when $f(x) = 0$ for $x < a$, $x > b$."

**19[Z].**—V. BUSH & S. H. CALDWELL, "A new type of differential analyzer," Franklin Institute, *J.*, v. 240, Oct. 1945, p. 255–326; 51 figures and illustrations.

The first proposal for a machine to solve differential equations appears to be due to Lord Kelvin, in 1876, but the present activity in the construction of such devices was initiated by the independent work of V. Bush, and others at the Massachusetts Institute of Technology, starting in 1925. The first machine capable of handling a fairly wide class of differential equations was described by Bush in 1931, and named the "Differential Analyzer." Since that time many similar machines have been built in this country, in England, Norway and Germany.

Engineers and physicists reduce many of their problems to differential equations, and families of particular solutions of those equations have proved to be of great service. The Massachusetts Institute of Technology and the Rockefeller Foundation joined forces in 1935 in an effort to produce a more flexible and accurate differential analyzer for the calculation of such solutions. The Differential Analyzer, which resulted from this program, was placed in service in 1942, and has been in use throughout the war. In November 1945, it was opened to the public for scientific work in general including that of industry.

The natural or canonical form of differential equation for the Analyzer differs somewhat from the forms usually chosen in the analytic study of differential equations. It may be written (the following notation is not used by the authors)

$$x_j = K_j \int_{x_{j0}}^{x_i} x_k dx_i + x_{j0},$$

$$x_p = A_{pq} x_q + C_p, \quad x_r = x_s + x_t + D_r, \quad x_u = f_u(x_v) + E_u,$$

where $i, j, k, p, q, r, s, t, u, v$ range over suitable restricted integral values. These equations represent the types of mechanism available for the solution.

The variables, $x$, are represented in the Analyzer by the angular positions of shafts, the shafts being continuously rotated during the solution of a problem. The first equation of the canonical set is "solved" by devices called "integrators," each of which consists of a wheel rolling on the surface of a rotating disk. It is interesting to note that the variable of integration may be any of the variables in the problem, and in any one problem, integrals may be taken with respect to several distinct variables. This fact is frequently relied upon to simplify equations.

The remaining equations of the canonical set are mechanized by means of gear boxes, (multiplication by constants $A_{pq}$), "adders," commonly known as differentials (addition) and function tables, on which graphs of the functions $f_u$ are plotted.

The constants, $C_p$, $D_r$, $E_u$ and $x_{j0}$ must be set into the machine at the start of each problem. Their magnitudes are assigned by the initial conditions of each required solution. They are introduced to the machine at scales which are selected with due regard to the ranges over which the variables will run during the solution, the ranges available in the equipment, and the accuracy with which the solution is desired.

The M.I.T. Analyzer represents an advance over previous analyzers to some extent in its increased accuracy, but to a far greater extent in the speed with which new problems can be set up. The accuracy of such complicated mechanisms is difficult to evaluate, even if an agreement can be reached on the definition of the term. Errors in any continuous computer, of which the Analyzer is an example, depend critically on the state of maintenance of the parts, on dirt, temperature, wear, etc. etc., as well as upon the wisdom used in the choice of scale factors and speeds of solution employed.

The errors in the solution depend, of course, upon the type of differential equation being solved, some equations being sensitive to inaccuracies in the elements and others being very stable.

In view of the facts noted, it is dangerous to quote any figures, but in order to give the uninformed a rough idea, it may be said that the errors in a trajectory are likely to be of the order of one part in ten thousand, with a very liberal factor one way or the other.

The speed with which new problems can be set up on the machine, once they have been reduced to the form of codes on punched tapes, is such that only about 15 minutes of machine time is required. This contrasts favorably with the one or two days required to set up the previous analyzers.

The class of differential equations which can be handled with the Analyzer is somewhat wider than that heretofore treated, because the number of computing elements—integrators, gear boxes, etc.—is greater. There are 18 integrators available. This does not ordinarily mean that 18th order equations can be solved, since some of the integrators are likely to be needed for the generation of special functions. For example, if a product $x_j x_k$ appears in the original differential equation, it is replaced by

$$x_j x_k = \int x_j dx_k + \int x_k dx_j,$$

which requires two integrators to generate.

In addition to such functions as may be defined by subsidiary differential equations, the Analyzer can accept problems involving three arbitrary functions of one variable each. There is no means for dealing with functions of two or more variables, unless these functions can be reduced to the solutions of subsidiary differential equations, or to functions of one variable. There are in existence plans for four "function units" which will accept values tabulated at discrete intervals and interpolate continuously between them. These units are not yet in use.

Results of the computation are presented either graphically or in the form of typed tables.

Special techniques and schematic symbolisms have been developed for use in reducing mathematics to machine connections and are described by Bush and Caldwell. The symbols are simple and suggest the mechanical units they are intended to represent. While not logically necessary, the schematic or short-hand notation is important practically, as an aid to the rapid and easy reduction of problems to a form acceptable to the analyzer.

The design and construction of continuous computers having the accuracy and flexibility of the M.I.T. Analyzer, demands a high order of ingenuity and mechanical skill. Some of the outstanding problems met and solved by the Massachusetts Institute of Technology group and their associates are described by Bush and Caldwell. Among these are the design of data-transmission and servo units, the development of codes and related equipment for the rapid set-up of problems, and the conversion of continuous variables (shaft rotations) into digital form, suitable for printing.

The M.I.T. Analyzer is finding and will continue to find a field of usefulness in the solution of the differential equations of engineering and physics, especially as engineers delve further into situations which demand the solution of non-linear relations.

G. R. STIBITZ

**20[Z].**—INTERNATIONAL BUSINESS MACHINES CORP., *IBM Automatic Sequence Controlled Calculator*, New York, I.B.M., 590 Madison Avenue, 1945. 6 p. (21.5 × 28 cm.) + folding plate (83 × 28 cm.). Copies may be procured on application to the publisher.

This publication contains some notes on the development of this electro-mechanical calculator installed and constantly working in the Cruft Memorial Laboratory at Harvard University; it was officially turned over to the University on August 7, 1944, and has been, and still is, exclusively used by the U. S. Navy. The folding plate gives a complete view of the front side of the machine, and various statistics concerning the number of elements in the machine and the material used in its construction.

**21[Z].**—WILLIAM E. MORRELL, "A slide rule for the addition of squares," *Science*, n.s., v. 103, 25 Jan. 1946, p. 113–114.

The author describes a slide rule which he constructed, with appropriate square and square-root scales, for evaluating such an expression as $d = (x^2 + y^2 + z^2)^{\frac{1}{2}}$.

## NOTES

For a Note on Harry Bateman see RMT **289**.

**51.** EARLY DECIMAL DIVISION OF THE SEXAGESIMAL DEGREE (see N **29**, p. 400f; N **41**, p. 454).—In our previous notes on this topic we listed 8 or 9 editions of *De Thiende*, 1585, by SIMON STEVIN; but we forgot to give a reference to the Dutch edition, 9 or 10, of EZECHIEL DE DECKER, *Eerste Deel van de Nieuwe Telkonst . . . Noch is hier achter byghevoeght de Thiende*

*van Symon Stevin van Brugghe.* Gouda, 1626. It occupies the last 27 p. of the volume and has its own title page.

R. C. A.

**52.** THE FUNCTION LN $F(\frac{1}{2}n, \frac{1}{2}, x)$ AND OTHER RELATED FUNCTIONS.—The NYMTP has carried out extensive computations on the function ln $F(\frac{1}{2}n, \frac{1}{2}, x)$ and other related functions to be mentioned below for $n = 1(2)201$ and $x = 0(.01).6(.1)2(.2)7(1)45(5)100$. The confluent hypergeometric function $F(\frac{1}{2}n, \frac{1}{2}, x)$ is the solution of the differential equation

$$2xF'' + (1 - 2x)F' - nF = 0, \tag{1}$$

satisfying the boundary condition $F(\frac{1}{2}n, \frac{1}{2}, 0) = 1$. If we put $F = e^{\frac{1}{2}x}x^{-\frac{1}{4}}v$, then it is readily found that $v$ satisfies the differential equation

$$16x^2v'' + [3 + (4 - 8n)x - 4x^2]v = 0;\ v(0) = 1.$$

For $n \gg x$, the Brillouin-Kramers-Wentzel approximation method yields

$$v \sim x^{\frac{1}{4}} \cdot e^{\sqrt{2nx}}, \text{ whence } F \sim e^{\frac{1}{2}x} \cdot e^{\sqrt{2nx}} \sim e^{\sqrt{2nx}},$$

and therefore ln $F \sim \sqrt{2nx}$. The function ln $F/\sqrt{2nx}$ is considerably smoother than either $F$ or ln $F$, and for this reason will be tabulated over the major range of $n$ and $x$.

For $x$ small, neither $F$ nor ln $F$ nor ln $F/\sqrt{2nx}$ differences well for fixed $n$ and running $x$, but the function $F/\cosh\sqrt{(2n-1)x}$ does; it is therefore contemplated to tabulate the latter function for small values of $x$. That $F \sim \cosh\sqrt{(2n-1)x}$ for small $x$ may be shown as follows: If in (1) we make the substitution $x = y^2$ then

$$\frac{d^2F}{dy^2} - 2y\frac{dF}{dy} - 2nF = 0.$$

If we put $F = e^{\frac{1}{2}y^2}u$, then

$$u'' + (1 - 2n - y^2)u = 0;\quad u(0) = 1. \tag{2}$$

For $x$ (and therefore $y$) small, (2) yields

$$u'' + (1 - 2n)u = 0;\quad u(0) = 1;$$

whence $u = \cosh\sqrt{2n-1}\,y$, and therefore $F \sim e^{\frac{1}{2}y^2}\cosh\sqrt{2n-1}\,y$ or $F \sim \cosh\sqrt{2n-1}\,y$.

ARNOLD N. LOWAN & WILLIAM HORENSTEIN

NYMTP

**53.** GUIDE (*MTAC*, no. 7), SUPPL. 3 (for Suppl. 1–2, see *MTAC*, v. 1, p. 403–404, v. 2, p. 59).—In *MTAC*, v. 1, p. 289, is listed C. C. FURNAS, "Evaluation of the modified Bessel function of the first kind and zeroth order," *Amer. Math. Mo.*, v. 37, 1930, with a table, p. 287, of $I_0(x)$, $x = [12(2)20(10)100(100)1000;\ 4S]$. On p. 492 of the same volume, S. A. SCHELKUNOFF gives a revised form of the table to 5S, from which the Furnas values differ by from $-0.149$ to $+0.111$ times the corresponding power of 10.

Since J. W. WRENCH, JR. had calculated $I_0(30)$ to 20S, *MTAC*, v. 1, p. 200, it was suggested to him that he might check the whole of Schelkunoff's table. He has now (26 Nov. 1945) made the following report: "I recalculated the entire table to 8 and 9S, and thereby proved Schelkunoff's values to be entirely free from error."

This calculation also enabled him to check 6 values [$x = 12(2)20, 30$] of $I_0(x)$ in the 6S table of H. E. H. WRINCH & D. M. WRINCH, *Phil. Mag.*, s. 6, v. 45, 1923, p. 847, and v. 47, 1924, p. 63. Compare *MTAC*, v. 1, p. 226. The values for $x = 12$ and 14 checked exactly; for $x = 16$ the last digit needs to be increased by unity; for $x = 18$ an increase of 3 units is necessary; for $x = 20$, a decrease of 3 units; and for $x = 30$ a decrease of 2 units.

R. C. A.

**54.** A HUGE NUMBER.—Introductory to comment on a recent paper involving tables, I shall reprint some of the items published in my note on "Huge Numbers," *Amer. Math. Mo.*, v. 28, 1921, p. 393–394. What is the largest number that we can express by three digits? One answer is $9^{(9^9)}$ or $9^{387420489}$. C. A. LAISANT drew attention to this number in his *Initiations Mathématiques*, Paris, 1906 (English edition, London, 1913, p. 101). He then remarks that in decimal numeration this number would have 369693100 figures. To write it on a single strip of paper, supposing that each figure occupied a space of one fifth of an inch, the length of the strip would be 1166 miles, 1690 yards, 1 foot, 8 inches.

Writing in 1913 A. C. D. CROMMELIN stated, Br. Astr. Assoc., *J.*, v. 23, p. 380–381, that he had come across the problem "in an old logarithm book." By the aid of 61-figure logarithms Crommelin found $\log N = 369{,}693{,}099.6315703587 \cdots$; whence the number of figures indicated above. He then found the first 28 figures of $N$ to be 428,124,773, 175,747,048,036,987,115,9, and the last three to be 289. Crommelin then continues, "A knowledge of 30 figures out of 300 million may seem trifling, but in reality the error involved in taking all the remaining figures as zeros is only one part in a thousand quadrillions.[1] If the number were printed with 16 figures to an inch (about the lightest packing for decent legibility), it would extend over 364.7 miles. . . . If printed in a series of large volumes we might get 14,000 figures to a page, and with 800 pages to the volume it would fill 33 volumes. There are more than twice as many digits in the number as there are letters in the whole of the *Encyclopædia Britannica*. . . . We may take it as morally certain that we can write with three digits a number vastly exceeding the number of electrons in the whole of creation, which is a somewhat startling fact. Indeed, even the number $4^{4^4}$ (which is 13407813 followed by 147 other figures) probably exceeds the number of electrons in creation."

In Br. Astr. Assoc., *J.*, v. 26, 1915, p. 46–47, D. G. MCINTYRE gave the last eight figures as 17,177,289.

We now turn to *Matematisk Tidsskrift*, A, 1941, p. 63–70, and an article by CHR. WEISS, entitled "'Hu', Tallet $9^{(9^9)}$ og Endecifrene i Potenser af 9." Weiss found $\log N$ to 74D as follows: 369693099.631570 358743 543095 095482 929683 400246 855547 962405 328896 342699 975257 003407 00; whence he derived the first 60 figures of $N$ to be (details of computation given) 4281 247731 757470 480369 871159 305635 213390 554822 414435 141747 53.

On p. 67 is a table of the last 26 figures for each of 35 values of $9^n$, $n = 89{,}100{,}200, \cdots, 387420489$. Thus the last 26 figures of $N$ are found to be

24 178799 359681 422627 177289.

These results check with those quoted above, except in the case of the first of the McIntyre figures. Weiss gives also two tables and formulae for finding last figures of $9^n$.

J. W. MEARES in Br. Astron. Assoc., *J.*, v. 31, 1921, p. 277–278, comments on $9!^{(9!)!}$ and finds that its value is greater than 10 to the power $10^{2000000}$ but less than 10 to the power $10^{2000001}$.

R. C. A.

[1] In accordance with British usage, Crommelin here means $1000 \times 10^{24}$; in the United States this would be interpreted as $1000 \times 10^{15}$.

**55.** A NEW RESULT CONCERNING A MERSENNE NUMBER.—(Compare N. **23**, **33**, v. 1, p. 333, 404). On 9 February 1946 I finished testing the character of the Mersenne number $M_{229} = 2^{229} - 1 = 8627\ 18293\ 34882\ 04734\ 29344\ 48278\ 46281\ 81556\ 38862\ 15212\ 98319\ 39531\ 55279\ 74911$. Since the final residue, the 228th, was not zero the conclusion is that $M_{229}$ is composite.

The Lucasian sequence used was 4,14,194,37634,1416317954, etc.

The 228th residue was found to be 1970 11660 94225 75309 56180 91126 86257 27776 96596 41856 06805 84362 68648 91891.

Thus, among these numbers $M_p$, up to and including $p = 257$, there are only three whose characters are unknown, namely: $p = 193{,}199{,}227$. There are, however, eleven $M_p$, known to be composite, but of which no factor is known.

I have begun a similar investigation of $M_{199}$.

HORACE S. UHLER

12 Hawthorne Avenue
Hamden 14, Connecticut

## QUERIES

**17.** TABLES FOR CIRCLES.—In O. G. GREGORY, *Mathematics for Practical Men*, London, 1825, p. 406, after "A Table of Circles, from which knowing the diameters, the areas, circumferences, and sides of equal squares are found," by GOODWYN (see MTE **81**), Gregory remarks that this table was "to supersede the necessity of consulting some erroneous tables of the areas, &c. of circles recently put into circulation." What author and publication are here indicated? The *English Catalogue* lists the following anonymous item issued in the following year: *Tables of Areas and Circumferences of Circles*, 3 parts, London, 1826.

R. C. A.

## QUERIES—REPLIES

**21.** BRIGGS' ARITHMETICA LOGARITHMICA (Q**7**, v. 1, p. 170).—In my library is a copy of this volume with the extra 12 pages containing the

logarithms of numbers [100001(1)101000; 14D] and the square roots of integers [1(1)200; 11D], with first differences in each case.

ALBERT E. LOWNES

16 Barberry Hill
Providence, Rhode Island

EDITORIAL NOTE: At present only four existing copies of these extra pages are known to us. Rather extensive inquiry suggests that the copy owned by Mr. Lownes may be the only one in America.

**22.** INTEGRAL AND FUNCTIONAL TABLES.—(Q15, p. 459).—With reference to tables of $\int_0^z e^{-t^2}dt$, I should like to draw attention to my paper "An approximate method for calculating heat flow in an infinite medium heated by a cylinder," Phys. So. London, *Proc.*, v. 56, 1944, p. 365, where I have given a rough chart of this integral $[2\pi^{-1}\int_0^{\omega e^{i\beta}} e^{-t^2}dt]$ for a complex argument [$\beta = 0(15°)90°$; $\omega = 0(.1).3(.2).7, 1$], together with formulae for computation in a number of cases.

S. WHITEHEAD

British Electrical and Allied Industries
Research Assoc., London, England

**23.** ROOTS OF THE EQUATION TAN $x = cx$ (Q8, v. 1, p. 203; QR10, p. 336; 17, p. 459).—The NDRC report, *Tables for Solutions of the Wave Equation for Rectangular and Circular Boundaries having Finite Impedance* (see *MTAC*, v. 1, p. 438–440) has (in a different notation) the first four roots (i.e. the first four branches of the functions) of the equations $\tan x = cx$, $\tanh x = cx$, $\cot x = cx$, and $\coth x = cx$, tabulated as functions of the complex variable $c$. We have obtained expansions of $x$ in powers of $c$ and $1/c$ (used in preparation of the above mentioned NDRC report) from which one can easily compute values on the other branches or improve the accuracy of the above tables. We also have theoretical material and other types of expansions for the regions about the singularities of the inverse functions. This material is in our possession at the NYMTP, but is as yet available only to certain Government agencies and activities.

A. P. HILLMAN & H. E. SALZER

## CORRIGENDA

V. 1

P. 205, l. 17, *for* $j_{2,s/\pi}$, *read* $j_{2,s}/\pi$.
P. 213, $A_2$ 38, l. 3, *add* $x = 0(.5)12$; $B_2$ 4, *for* 5D, *read* 4D.
P. 214, $B_2$ 13, *for* $e^{x_s 2\theta}$, *read* $e^{-x_s 2\theta}$, and *for* 4, *read* .4; $B_2$ 17, *for* $J_1(\frac{1}{2}x_s)e^{-x_s 2t}$ *read* $J(\frac{1}{10}x)e^{-x_s 2}$.
P. 215, $B_6$ 8, *add* also 5D, $= 1(1)8$; $B_7$ 22, *for* =, *read* or.
P. 217, $C_2$ 7, delete $\leq$ 14D; $D_1$ 7, l. 3, when $n = 10, 11$, *read* $s = 3, 3$.
P. 221, $A_4$ 1, delete $x > 21.6$; $A_5$ 1, *for* $x > 21.6$, *read* for each $n$; $A_5$ 6, 7, *for* $Y_0(x)$, $Y_1(x)$, *read* $Y^{(0)}(x)$, $Y^{(1)}(x)$, and *transfer* these two entries as $C_2$ 8, 9.
P. 222, $A_5$ 12, *for* zeros, *read* first zero.
P. 226, $A_1$ 12, 13, *for* 5D, *read* 5S, and *for* 4D, *read* 4S.
P. 227, $A_2$ 28, *for* OKAYA 2, $x$, *read* OKAYA 2, $x^{-1}$
P. 241, B 5, *for* $J_{-1/6}(x)$, *read* $J_{-1/6}(\frac{1}{2}x^2)$; and *for* 41.035, *read* 41.305.
P. 245, l. 14, *for* $E_n(x)$, *read* $\mathbf{E}_n(x)$.
P. 251, B 16, *read* 4D, $Ki_2(x)$, SEELIGER, $x = .25(.01)1(.05)7$.
P. 284, l. 13, *for* $Y_n(x)$, for, *read* $Y_n(x)$ for each $n$, for.

P. 289, l. 4, *for* 393, *read* 395.
P. 414, l. 8, *for* 1942, *read* 1941.
P. 476, Riche de Prony, *for* Caspard, *read* Gaspard.

V. 2

P. 6, l. −8, *for* $e^{2z}$, *read* $e^{-2z}$.
P. 9, l. 11, *for* z or, *read* or $z_r$.
P. 11, l. −21, *for* St., *read* 51.
P. 15, l. 1, for *Tablitsy*, read *Tablitsy*.
P. 16, l. 32, 4.7, *for* Sums of Products, *read* Sums and Products; l. −11, *for* Gennochi's, *read* Genocchi's.
P. 29, l. 32, *for* give complex roots, *read* give complex roots directly as decimals.
P. 60, l. −1, *for* arccos(sinhy), *read* arccos(−y/sinhy).

hi's,

CLA

A.
B.
C.
D.
E.

F.

G.

H.
J.

I.
K.

L.
M.

N.
O.

P.

Q.

R.
S.
T.
U.

Z.

The
of the
Execut

Sub
bers ea
12, $1.
Constit
sellers.

## CLASSIFICATION OF TABLES, AND SUBCOMMITTEES

A. Arithmetical Tables. Mathematical Constants
B. Powers
C. Logarithms
D. Circular Functions
E. Hyperbolic and Exponential Functions
Professor DAVIS, *chairman*, Professor ELDER, Professor KETCHUM, Doctor LOWAN

F. Theory of Numbers
Professor LEHMER

G. Higher Algebra
Professor LEHMER

H. Numerical Solution of Equations
J. Summation of Series

---

I. Finite Differences. Interpolation
K. Statistics
Professor WILKS, *chairman*, Professor COCHRAN, Professor EISENHART

L. Higher Mathematical Functions
M. Integrals

---

N. Interest and Investment
O. Actuarial Science
Mister ELSTON, *chairman*, Mister THOMPSON, Mister WILLIAMSON

P. Engineering

---

Q. Astronomy
Doctor ECKERT, *chairman*, Doctor GOLDBERG, Miss KRAMPE

R. Geodesy
S. Physics
T. Chemistry
U. Navigation

---

Z. Calculating Machines and Mechanical Computation
Professor CALDWELL, *chairman*, Doctor COMRIE, *vice-chairman*
Professor AIKEN, Professor LEHMER, Doctor MILLER, Doctor STIBITZ, Professor TRAVIS, Mister WOMERSLEY

## EDITORIAL AND OTHER NOTICES

The addresses of all contributors to each issue of *MTAC* are given in that issue, those of the Committee being on cover 2. The use of initials only indicates a member of the Executive Committee.

Subscription price for each *calendar* year $3.00, *payable in advance;* ordinary single numbers each $1.00 (special number 7, *Guide to Tables of Bessel Functions*, $1.75; special number 12, $1.50), payable to the Division of Physical Sciences, National Research Council, 2101 Constitution Avenue, Washington, D. C. No reductions are made to Libraries or to Booksellers. No sample copies are distributed.

CONTENTS

APRIL 1946

Conference on Advanced Computation Techniques ..... 65

Recent Mathematical Tables ..... 68

**275** (Pederson); **276** (Kolmogorov); **277** (Great Britain); **278** (Dunoyer); **279** (Beeger); **280** (Gloden); **281** (Thébault); **282–284** (Salzer); **285** (Burrau); **286** (Grimsey); **287** (Corrington); **288** (Cossar & Edélyi); **289** (Isaacs); **290–291** (NYMTP); **292** (Bordoni); **293** (Great Britain); **294** (Heisenberg); **295** (Hamburg); **296–297** (Hydrographic Office, Tokyo); **298** (Weems)

Mathematical Tables—Errata ..... 85

**79** (Barlow-Comrie); **80** (Burgess); **81** (Goodwyn); **82** (NYMTP); **83** (Nielsen, Seeling, Patz)

Unpublished Mathematical Tables ..... 88

**42** (Beeger); **43** (Gupta); **44** (Hammer); **45** (Salzer)

Mechanical Aids to Computation ..... 89

**18** (Born, Fürth & Pringle); **19** (Bush & Caldwell); **20** (IBM); **21** (Morrell)

Notes ..... 91

**51** Early Decimal Division of the Sexagesimal Degree; **52** The Function Ln $F_1(\frac{1}{2}n, \frac{1}{2}, x)$ and other Related Functions; **53** *Guide*, Suppl. 3; **54** A Huge Number; **55** A New Result Concerning a Mersenne Number

Queries ..... 94

**17** Tables for Circles

Queries—Replies ..... 94

**21** Briggs' *Arithmetica Logarithmica;* **22** Integral and Functional Tables; **23** Roots of the Equation $\tan x = cx$

Corrigenda ..... 95

LANCASTER PRESS, INC., LANCASTER, PA.

65
68
85
88
89
91
94
94
95